

An Introduction to Complex Analysis

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## Preface

The purpose of this paper is to present an introductory survey of the basic ideas and concepts which form a foundation for the analytic study of the single-valued functions of a complex variable. In no way does it purport to be an in depth, rigorous, or complete analytic study. In it, however, many of the basic tools which are prerequisite to any meaningful study of complex functions are presented.

In chapter one, the idea of a function of a complex variable is developed. The linear fractional transformation is presented in some detail for the purpose of demonstrating some basic geometric concepts, and to introduce a few fundamental ideas used in the study of conformal or isogonal mappings. In the remainder of the chapter, some of the common functions, such as the power and root functions and some transcendental functions are discussed, but in considerably less detail than is the linear fractional transformation. The discussion of these latter functions is mainly from an algebraic view point.

Chapter two introduces the concept of an analytic function of a complex variable. The limit, continuity, and derivative of a single-valued complex function are defined. From these basic definitions, the fact that the Cauchy-Riemann conditions are necessary for the existence of a unique derivative is derived. It is then shown that under certain conditions, the Cauchy-Riemann conditions are not only necessary but also sufficient for the existence of a unique derivative.

In chapter three, analytic functions are again the main topic of discussion. In this chapter, however, the discussion centers mainly around the concept of the complex integral. The integral of a complex function is defined

and its direct relationship to the line integral in the two dimensional real plane is established. From this relationship Cauchy's integral theorem for simply connected regions is developed. The theorem is then extended to multiply connected regions. With the aid of the extended theorem Cauchy's integral formula is arrived at. To conclude chapter three, it is then shown, by using the integral formula, that if a function is analytic it has derivatives of all orders.

The prerequisites for the study of this paper are very few. An understanding of the complex numbers, their field axioms, the complex arithmetic operations of addition, subtraction, multiplication, and division along with the plane geometric interpretations of these is essential. Some familiarity with the complex number plane is also necessary. Along with these the reader should understand the calculus of real variables up through some basic ideas on partial derivatives. Some concepts which may not be immediate to the reader but which are essential to the understanding of the context of the paper are developed in four supplementary appendixes. The titles of these are: Regions; The Stereographic Projection; DeMoivre's Theorem; and Line Integrals.

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## I. Functions of a Complex Variable

Defn. 1. If  $z$  is any complex number in a region  $R$  (see Appendix I) of the complex plane, we call  $z$  a complex variable.

Defn. 2. If a complex variable  $w$  is so related to  $z$  such that for every  $z$  in  $R$ , there corresponds a definite value or set of values of  $w$ , then  $w$  is a function of the complex variable  $z$  (P. 2, 1).

If  $z=x+iy$ , then the usual notation is as follows:

$$w = u(x,y) + iv(x,y) = f(z)$$

where  $u$  and  $v$  are real functions of the real variables  $x$  and  $y$ . If  $w$  has one and only one value corresponding to each value of  $z$  in  $R$ , then  $f$  is called a single-valued function. If, on the other hand,  $w$  takes two or more values for some or all of the  $z$ 's in  $R$ , then  $f$  is called a multiple-valued function of  $z$ .

There are several types of functions of a complex variable.<sup>1</sup> The first type is the rational integral function and is of the form:

$$w = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$$

The  $a_i$ 's are complex constants and  $n$  is a positive integer. If  $a_n \neq 0$ , then the function is said to be of degree  $n$ .

A complex function which is the quotient of two rational integral functions is called a rational fractional function. It has the form:

$$w = \frac{a_n z^n + a_{n-1} z^{n-1} + \dots + a_0}{b_m z^m + b_{m-1} z^{m-1} + \dots + b_0}$$

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<sup>1</sup>In the context of this paper the word function shall be used to mean a single-valued function unless otherwise stated.

where  $m$  and  $n$  are positive integers which may or may not be equal. The maximum of  $m$  and  $n$  is called the degree of the function provided that the appropriate  $a_n$  or  $b_m$  does not equal 0. If  $m = n$  then either is the degree of the polynomial.

Any function which does not fall under either of the above classifications is called an irrational function.

An algebraic function is a function which is of the form:

$$f_0(z)w^n + f_1(z)w^{n-1} + f_2(z)w^{n-2} + \dots + f_n(z) = 0$$

where the  $f_i$ 's are rational integral functions, and the left hand side is irreducible, that is, it can not be expressed as a product of two polynomials neither being constant. It can be shown that all rational functions are algebraic functions.

Any function which is not algebraic, is said to be transcendental. This class would, of course, include all of the trigonometric, exponential, and logarithmic functions (P.22-3, 2).

In this section we will consider a particular complex function, the linear fractional transformation. The general form is as follows:

$$w = \frac{az + b}{cz + d}$$

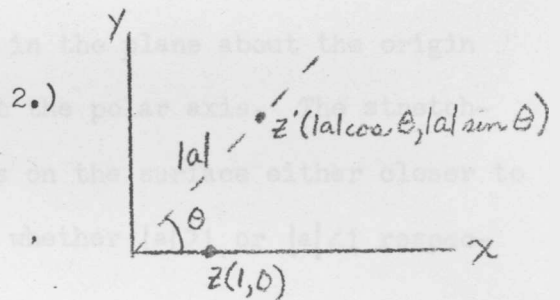
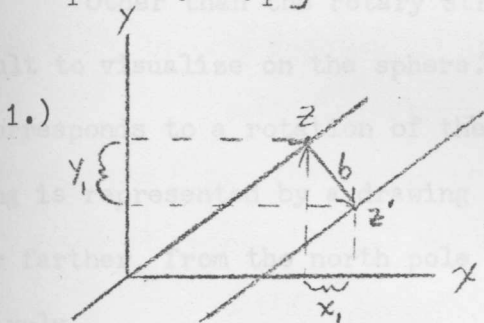
where  $a$ ,  $b$ ,  $c$ , and  $d$  are complex constants.

When dealing with a function in the real field such as  $y = f(x)$  it is sufficient to consider its behavior by plotting  $y$  versus  $x$  in the  $xy$ -plane. In the complex field, however, two planes are necessary for such an analysis. We call one plane the  $z$ -plane. It is in this plane that the independent variable,  $z = x+iy$ , is plotted. In the other, the  $w$  or image plane, the values corres-

ponding to the  $z$  values are plotted. Similarly, Riemann spheres (see Appendix II) can be used in the same manner. In many cases, as will be seen, it is convenient to consider the two planes (spheres) as coincident. We then say that the mapping is from the  $z$  plane (sphere) on itself.

To analyze the linear fractional transformation, we will first consider special cases and then generalize to try to get an understanding of the function. The first case to be considered is when  $c = 0$  and  $d = 1$ . Functions of this form,  $w = az + b$ , are called entire linear functions.

Let  $a = 1$  such that  $w = z + b$ . Now if  $b = 0$ , the mapping is nothing more than the identity, that is, any image and its preimage coincide. If  $b \neq 0$ , then the mapping becomes a translation or parallel displacement in the plane. Take, for example, the set  $\{z = x + iy : y = x\}$  and let  $b = x_1 + iy_1$  where  $x_1$  and  $y_1$  are constant. Then this set maps into the set  $\{z' : z' = (x + x_1) + i(y + y_1)\}$  (see figure 1). In this mapping, any particular image



$z'$  can be found by adding the vector  $b$  onto the radius vector  $z$ . The terminal point of the vector  $b$  will be the image point  $z'$  of  $z$ . The two sets above are said to be congruent.

Now let  $b = 0$  such that  $w = az$  and let  $a \neq 0$ . This mapping, according to the geometric interpretation of multiplication, is nothing more than a rotary stretching with center at the origin. That is, in order to find the image of  $z$ , rotate the point  $z$  through a positive angle equal to the amplitude of  $a$  and stretch it by the ratio  $1:|a|$  where  $|a|$  is the modulus of  $a$ .

For example, if  $z = 1$ , then  $z' = |a| \cos\theta + i|a| \sin\theta$  where  $\theta$  is the amplitude of  $a$  (see figure 2 on the previous page). If  $|a| = 1$  then the mapping is a simple rotation and is then a congruence mapping. If  $\theta = 0 + 2n\pi$  ( $n$  an integer) then the mapping is a pure stretching. This is a similarity mapping, and the ratio of similitude is  $|a|:1$ . For any arbitrary  $a$ , the mapping is a similarity mapping with the origin as center. Note also that rotating and stretching are commutative operations and can be carried out in either order.

Finally, let  $w = az + b$ , where  $a \neq 0$ . This mapping represents a rotary stretching and a translation. Or, noting that  $w = a(z + b/a)$ , it can be interpreted as a translation followed by a rotary stretching. The translating vectors are not equal in the two cases. In the second case the rotation is about the point  $b/a$  since this point is mapped into the origin by the translation.

Other than the rotary stretching, the preceding mappings are difficult to visualize on the sphere.<sup>2</sup> A rotation in the plane about the origin corresponds to a rotation of the sphere about the polar axis. The stretching is represented by a drawing of the points on the surface either closer to or farther from the north pole according to whether  $|a| > 1$  or  $|a| < 1$  respectively.

Now consider the fractional linear function:

$$w = 1/z$$

To facilitate the analysis of this function we express the variables in polar form.

$$z = p(\cos\theta + i\sin\theta) \text{ and } w = p'(\cos\mathbb{D} + i\sin\mathbb{D}).$$

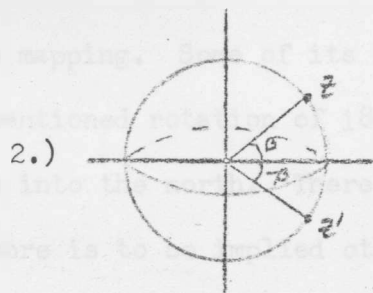
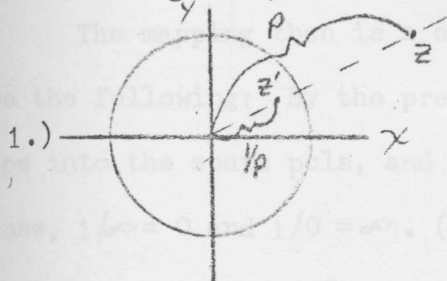
Then according to DeMoivre's theorem (see Appendix III)  $p' = 1/p$  and  $\mathbb{D} = -\theta$ .

<sup>2</sup>In the context of this work the word sphere shall always refer to the Riemann sphere.

Therefore, the mapping from  $z$  to  $w$  can be accomplished in two steps:

1. a mapping from a point  $z$  to a point  $z'$  which has the same amplitude but a reciprocal modulus
2. a mapping from the point  $z'$  to a point  $w$  which has the same absolute value but a negative amplitude.

The first step on the plane represents a reflection in the unit circle (see figure 1). On the sphere it represents a reflection about the equatorial plane (see figure 2).



On the plane the idea is easily understandable. For the sphere the following will explain the concept. Recall that on the sphere the following properties hold (see Appendix II):

$$\cot(\pi/4 - \beta/2) = p \quad \text{and} \quad \cot(\pi/4 - \beta'/2) = 1/p$$

where  $\beta$  and  $\beta'$  are the angles of latitude<sup>3</sup>, and  $p$  and  $1/p$  are the amplitudes of  $z$  and  $z'$  respectively. But since  $p \cdot 1/p = 1$ ,

$$\cot(\pi/4 - \beta/2) \cot(\pi/4 - \beta'/2) = 1$$

and therefore,  $\beta + \beta' = 0$ , or  $\beta = -\beta'$ . Also recall that the unit circle maps into the equator of the Riemann sphere (see Appendix II) and therefore, it is reasonable to assume that a reflection with respect to the unit circle in the plane would correspond to a reflection about the equatorial plane on the sphere.

The second step represents simply the passage from a number to its

<sup>3</sup>The angle of latitude is the angle of the point either above(positive) or below(negative) the equatorial plane on the Riemann sphere.

conjugate. In the plane, this is simply a reflection with respect to the line of the real axis. The mapping on the sphere from  $z$  to  $w$  can be accomplished then in two steps. First by reflection with respect to the plane of the equator, and second by reflection with respect to the plane of the prime meridian.<sup>4</sup> These two, however, can be accomplished in one step,- a reflection with respect to the line of the intersection of the two planes, a rotation of the sphere through  $180^\circ$  about this axis which passes through the points  $z = 1$  and  $z = -1$ .

The mapping then is a congruence mapping. Some of its characteristics are the following: By the previously mentioned rotation of  $180^\circ$  the north pole maps into the south pole, and the south into the north. Therefore, in this sense,  $1/\infty = 0$  and  $1/0 = \infty$ . (Nothing more is to be implied other than the point  $0$  maps to  $\infty$  and  $\infty$  maps to  $0$ .) If we accept this, the function is one-to-one and maps the two full spheres onto each other. Every point with the exceptions of  $z = 1$  and  $z = -1$  maps to a distinct point different from itself. The points  $z = 1$  and  $z = -1$  map into themselves and are called the fixed points of the mapping.

Again note that the mapping  $w = 1/z$  can be broken into two parts:

1. a mapping from a point  $z$  to a point  $z'$  which has the same amplitude but a reciprocal modulus
2. a mapping from the point  $z'$  to a point  $w$  which has the same absolute value but a negative amplitude.

The first mapping, which is called reflection in the unit circle, has an important property which will be discussed here. Letting the point  $z$  have the

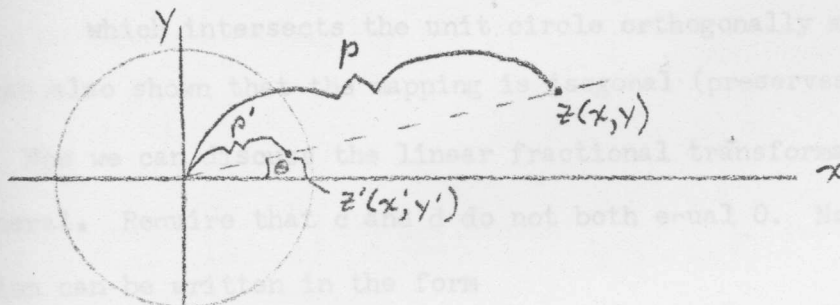
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<sup>4</sup>The prime meridian on the sphere is the stereographic image of the positive real axis of the complex plane.

coordinates  $(x, y)$  and the point  $z'$  the coordinates  $(x', y')$  and each having a radial distance of  $p$  and  $p'$  respectively, the following relations hold (see figure below):

$$\cos\theta = x/p = x'/p' \quad \text{and} \quad \sin\theta = y/p = y'/p'$$

where  $\theta$  is the amplitude of  $z$  and  $z'$  which remains constant under the mapping.



But  $p^2 = x^2 + y^2$ ,  $p'^2 = x'^2 + y'^2$ , and  $pp' = 1$ , so then

$$x' = \frac{x}{x^2 + y^2} \quad \text{and} \quad y' = \frac{y}{x^2 + y^2}$$

and also

$$x = \frac{x'}{x'^2 + y'^2} \quad \text{and} \quad y = \frac{y'}{x'^2 + y'^2}$$

Every circle or line in the plane has an equation of the form

$$\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0$$

where  $\alpha, \beta, \gamma, \delta$  are real constants. Under the mapping the image has an equation of the form

$$\alpha + \beta x' + \gamma y' + \delta(x'^2 + y'^2) = 0$$

which represents a line or a circle. Therefore, a line maps into a line or a circle, and a circle maps into a line or a circle under the mapping. Such a mapping is said to be circular (see also Appendix II). From this the mapping  $w = 1/z$  then has the following properties.

- a. A straight line which does not pass through the origin maps into a circle through the origin.
- b. A straight line passing through the origin maps into itself.
- c. A circle which passes through the origin maps into a straight line

not passing through the origin.

- d. A circle which passes through the unit circle maps into a circle which intersects the unit circle in the same point (points) as its preimage. More particularly, a circle (straight lines included) which intersects the unit circle orthogonally maps into itself.

It can also be shown that the mapping is isogonal (preserves angles).

Now we can discuss the linear fractional transformation  $w = \frac{az + b}{cz + d}$  in general. Require that  $c$  and  $d$  do not both equal 0. Noting that the function can be written in the form

$$I.) \quad w = -\frac{ad - bc}{c} \frac{1}{cz + d} + \frac{a}{c}$$

it can be seen that the mapping can be accomplished in three steps:

$$1.) \quad z' = cz + d$$

$$2.) \quad z'' = 1/z'$$

$$3.) \quad w = a_1 z'' + b_1$$

where  $a_1 = -\frac{ad - bc}{c}$  and  $b_1 = a/c$

Then according to equation I,  $\frac{az + b}{cz + d} = \frac{a}{c}$ . Since steps 1 and 3 are similarity mappings, and as was shown before, the mapping  $w = 1/z$  maps the whole  $z$  plane (sphere) onto the whole  $w$  plane (sphere) it can now be stated that the linear fractional transformation is a one-to-one mapping of the full  $z$  plane (sphere) onto the full  $w$  plane (sphere). It can also be stated by previous demonstration that the mapping is circular. The last statement is true because steps one and three are similarity mappings and step two as discussed previously is a circular mapping.

Since the linear fractional transformation  $w = \frac{az + b}{cz + d}$  is one-to-one and onto the inverse transformation exists and is of the form  $z = \frac{-dw + b}{cw - a}$  which is obtained simply by solving for  $z$  in the original mapping. But this is also a linear fractional transformation.

Putting the coefficients of the transformation in a matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  the following properties are immediate. If  $z = \frac{mz + n}{oz + p}$  and  $w = \frac{rz + s}{tz + u}$  it can be shown that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} m & n \\ o & p \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} mr + nt & ms + nu \\ or + pt & os + pu \end{pmatrix}$$

There also exists an identity

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and every element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  has an inverse  $\begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$ . Multiplication of ma-

trices is associative, and therefore, the set of linear fractional transformations with multiplication as defined form a group.

Now consider the fixed points of the mapping. They are the points such that  $z = \frac{az + b}{cz + d}$ . From this the equation  $cz^2 - (a - d)z - b = 0$  follows. Note that if  $c = 0$ ,  $b = 0$  and  $(a - d) = 0$  the transformation is simply the identity mapping  $w = z$ . Therefore, it is seen immediately that if a mapping has more than two fixed points it must be the identity mapping because only when the coefficients of the above quadratic are all 0 can there be more than two fixed points.

Recall that under the linear fractional transformation  $\infty$  maps into the point  $a/c$ . Therefore, both fixed points of this mapping (which may coincide) are finite. If the mapping is entire (of the form  $w = az + b$ ) then at least one of the fixed points is infinity. More especially, if  $w = z + b$  then infinity is the only fixed point since the mapping is nothing more than a translation. Therefore the linear mapping is entire if and only if infinity is a fixed point, and infinity is the only fixed point if and only if the mapping is a translation.

Further insight and understanding about the linear fractional transfor-

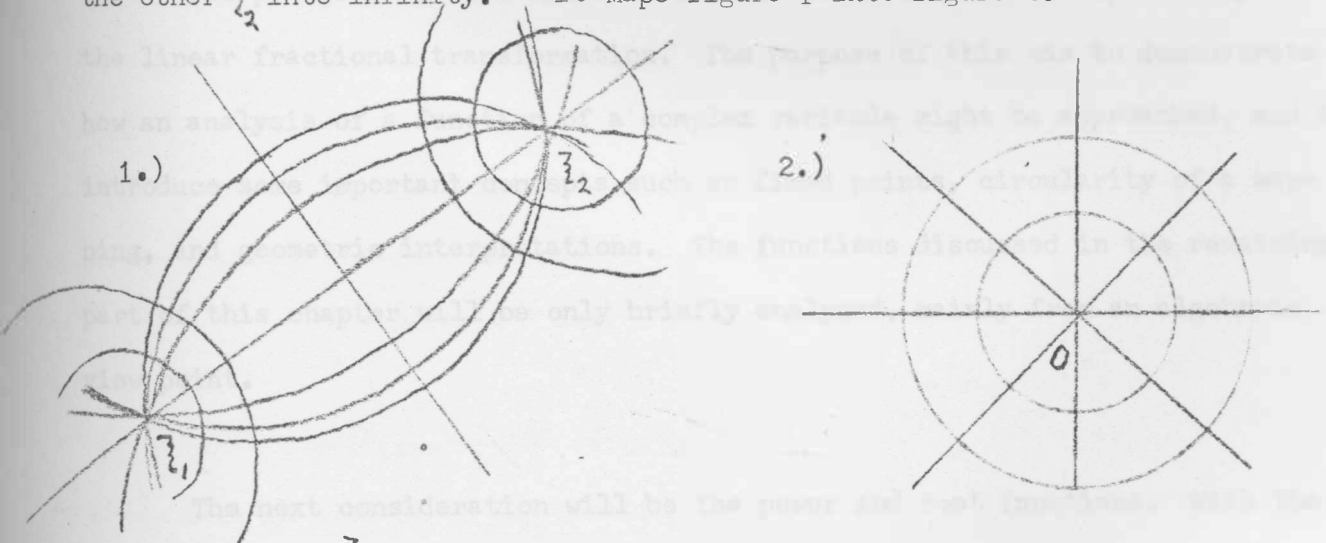
ation can be obtained through the following fixed point analysis. Take the function  $w = az + b$ . Now this mapping has the fixed point infinity from the argument above. But if  $z = az + b$  and it is required that the mapping not be a translation ( $a \neq 1$ ) then the other fixed point is  $z = b/(1 - a)$ , call this point  $\xi$ . By algebraic manipulation the entire mapping can be written

$$w - \xi = a(z - \xi)$$

The geometric interpretation is as follows. A translation  $z - \xi$  is effected in the  $z$  plane. The  $z$  plane is then mapped to the  $w$  plane by a rotary stretching. Since the translation moves the point  $\xi$  to the origin the function is interpreted to be a rotary stretching about the point  $\xi$ .

Now consider the general linear fractional function  $w = \frac{az + b}{cz + d}$  where  $c \neq 0$ . This mapping has two fixed points call them  $\xi_1$  and  $\xi_2$ . Since the mapping is circular, the set of all circles passing through the points  $\xi_1$  and  $\xi_2$  will map into itself. This can be demonstrated by a fixed point analysis also.

Let  $Z = \frac{z - \xi_1}{z - \xi_2} = g(z)$ . This maps one fixed point  $\xi_1$  into the origin, and the other  $\xi_2$  into infinity. It maps figure 1 into figure 2.



Also let  $W = \frac{w - \xi_1}{w - \xi_2} = g(w)$  which acts the same in the  $w$  plane. But since the set of all linear transformations form a group, the mapping  $w = f(z) = \frac{az + b}{cz + d}$  can be effected in the following manner:

$$w = g^{-1}(W) = f(g^{-1}(Z))$$

Now premultiply both sides of the equation by  $g$  to get

$$1) \quad W = gg^{-1}(w) = gfg^{-1}(Z)$$

where  $fg(z)$  is interpreted to mean  $f(g(z))$ . In effect, the function 1 above maps the  $Z$  plane into the  $z$  plane, the  $z$  plane into the  $w$  plane, and finally the  $w$  plane into the  $W$  plane. But  $g^{-1}(0) = z_1$ ,  $f(z_1) = z_1$ , and  $g(z_1) = 0$ . The same can be done with the point infinity to show that  $gfg^{-1}(\infty) = \infty$ . Consequently, 0 and  $\infty$  are the only fixed points of the mapping  $W = gfg^{-1}(Z)$  and the function is therefore, a rotary stretching with center at the origin. Recall that under the mapping  $g$ , the set of circles through the points  $z_1$  and  $z_2$  is transformed into the set of straight lines through the origin. If  $f$  is effected in the  $Z$  and  $W$  planes a rotation occurs as was demonstrated above. Mapping the  $Z$  and  $W$  planes back to the original  $z$  and  $w$  planes it is seen then that the set circles through the points  $z_1$  and  $z_2$  are unaffected by the transformation  $f$  since in the  $Z$  and  $W$  planes the rotation does not change the set of lines passing through the origin. (P. 42-57, 3)

The previous section has been devoted to a somewhat in depth study of the linear fractional transformation. The purpose of this was to demonstrate how an analysis of a function of a complex variable might be approached, and to introduce some important concepts such as fixed points, circularity of a mapping, and geometric interpretations. The functions discussed in the remaining part of this chapter will be only briefly analyzed, mainly from an algebraic view point.

The next consideration will be the power and root functions. With the use of DeMoivre's theorem (see Appendix III) this analysis becomes quite simple. DeMoivre's theorem states:

$$z^n = (p(\cos\theta + isin\theta))^n = p^n(\cos n\theta + isin n\theta)$$

for all  $n$  an element of the real numbers, where  $p$  is the modulus and  $\theta$  is the amplitude of  $z$ . Therefore, the interpretation of the power and root functions is immediate. We note also that if  $n = 0$

$$w = p^0(\cos 0 + i \sin 0) = 1$$

as would be hoped.

Another interesting point that might be made is the following:

$$z^{-n} = \frac{1}{p^n}(\cos -n\theta + i \sin -n\theta) = \frac{1}{p^n}(\cos n\theta - i \sin n\theta)$$

$$= \frac{p^n}{p^{2n}}(\cos n\theta - i \sin n\theta)$$

But,  $p^n(\cos n\theta - i \sin n\theta)$  is the complement of  $z^n$  which will be denoted by  $\overline{(z^n)}$ . Therefore,

$$z^{-n} = \frac{\overline{(z^n)}}{z^{2n}}$$

where  $z^{2n} = p^{2n}$ .

Now we have the tools which will allow us to find the  $n$   $n^{\text{th}}$  roots of any complex number. Noting that

$$\cos n\theta = \cos(n\theta + 2\pi k) = \cos n\left(\theta + \frac{2\pi k}{n}\right)$$

and

$$\sin n\theta = \sin(n\theta + 2\pi k) = \sin n\left(\theta + \frac{2\pi k}{n}\right)$$

where  $k$  is any integer. We have then that

$$z = p(\cos(\theta + 2\pi k) + i \sin(\theta + 2\pi k))$$

and

$$z^{1/n} = p^{1/n} \left( \cos \frac{\theta + 2\pi k}{n} + i \sin \frac{\theta + 2\pi k}{n} \right)$$

where  $n$  is an element of the positive integers and  $k = 1, 2, 3, \dots, n$ . From this the  $n$   $n^{\text{th}}$  roots of  $z$  can be obtained. For example, to find the four fourth roots of  $-1$

$$\theta = \quad \quad \quad k = 1, 2, 3, 4. \quad \quad \quad p = 1$$

$$\text{for } k = 1 \quad \sqrt[4]{-1} = 1(\cos 3\pi/4 + i \sin 3\pi/4)$$

$$= -\sqrt{2}/2 + i\sqrt{2}/2$$

$$\text{for } k = 2 \quad \sqrt[4]{-1} = 1(\cos 5\pi/4 + i\sin 5\pi/4)$$

$$= -\sqrt{2}/2 - i\sqrt{2}/2$$

$$\text{for } k = 3 \quad \sqrt[4]{-1} = 1(\cos 7\pi/4 + i\sin 7\pi/4)$$

$$= \sqrt{2}/2 - i\sqrt{2}/2$$

$$\text{for } k = 4 \quad \sqrt[4]{-1} = 1(\cos 9\pi/4 + i\sin 9\pi/4)$$

$$= \sqrt{2}/2 + i\sqrt{2}/2$$

Up to this point, a few of the elementary algebraic functions have been considered. The question that naturally arises now is whether or not reasonable definitions for some of the transcendental functions such as the trigonometric, exponential, and hyperbolic functions can be arrived at. Recalling that the transcendental functions in the real field can be expressed by means of infinite power series of the real variable, it can be shown that the complex functions are also expressible by means of infinite power series of the complex variable although such a demonstration would be beyond the scope of this paper. Just as is the case with real functions the domain of convergence of the series must be analyzed in the complex case. This, however, will be left to further study. A few of the basic power series definitions of the transcendental functions are as follows:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$

$$\sinh z = 1/2(e^z - e^{-z})$$

$$\cosh z = 1/2(e^z + e^{-z})$$

From these definitions the elementary identities can be derived. Some examples are:

$$\cos z = 1/2(e^{iz} + e^{-iz}) = \cosh iz$$

$$i \sin z = 1/2(e^{iz} - e^{-iz}) = \sinh iz$$

$$\sin^2 z + \cos^2 z = 1$$

$$\cosh^2 z - \sinh^2 z = 1$$

and so on.<sup>5</sup> (P. 681-6, 4)

In this chapter an attempt has been made at the explanation of the idea of a function of a complex variable. Hopefully, it is realized at this point that there are many ways in which the functions of a complex variable are similar to the functions of a real variable, and also that there are many ways in which they are different. The next topic of discussion is the analytic functions. This will be the purpose of the next chapter.

<sup>5</sup>For further reference on infinite power series of a complex variable and the transcendental functions see Knopp, reference 3, chapters VII, VIII, XI, XII, and XIII.

## II. Analytic Functions

If  $z = x + iy$ ,  $x \rightarrow x_0$ , and  $y \rightarrow y_0$ , then it can be said that  $z \rightarrow z = x_0 + iy_0$ . Now if  $f(z)$  is a single-valued function defined in some sufficiently small circular region with  $z_0$  as center (neighborhood of  $z_0$ ) then  $f(z)$  has a limit at  $z_0$  if as  $z$  approaches  $z_0$  from any direction in the  $z$  plane,  $f(z)$  approaches a definite value  $L$  in the image plane. The usual notation is

$$\lim_{z \rightarrow z_0} f(z) = L$$

The  $\epsilon, \delta$  form of the definition of a limit of  $f(z)$ , having the above requirements, would be as follows: For every  $\epsilon$ , an arbitrarily small but fixed positive value, there exists a  $\delta$  such that  $|f(z) - L| < \epsilon$  whenever  $0 < |z - z_0| < \delta$ . Although the bars in the complex field have a different meaning than do those in the real field, it can be seen that the formal limit definitions for both are identical and consequently, all the limit theorems in the real field can be similarly proven in the complex field. Furthermore, if  $f(z_0)$  exists, if the limit as  $z$  approaches  $z_0$  exists, and if  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$ , then  $f(z)$  is said to be continuous at  $z_0$ . (P. 538, 5)

To say that a function of a complex variable is differentiable at a point  $z_0$ , it is meant that the following limit exists

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

Again note that this definition is identical to its counterpart in the real field. Because of this, all the rules of differentiation are the same as those in the real field. The complex derivative is, however, different from the real derivative in that the limit concept is somewhat different. Recall that in the real field,  $h$  was either a positive or a negative value. In the complex field,  $h$  can be any complex number which must approach 0. That is to say that  $z_0 + h$  may approach  $z_0$  along any curve ending at  $z_0$ . Furthermore, if

if a function is differentiable at a point, it is continuous there. The proof of this is also approached in the same manner as in the real field. That is as follows: Since the limit exists (All the limits below are taken as  $h$  approaches 0 unless otherwise specified.)

$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$  a finite number. But by the basic limit theorem for quotients we have

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{\lim_{h \rightarrow 0} (f(z_0 + h) - f(z_0))}{\lim_{h \rightarrow 0} h} = f'(z_0)$$

and

$$\lim_{h \rightarrow 0} (f(z_0 + h) - f(z_0)) = \lim_{h \rightarrow 0} h \cdot 0 = 0$$

$$\lim_{h \rightarrow 0} f(z_0 + h) = \lim_{h \rightarrow 0} f(z_0)$$

or

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

Therefore, differentiability at a point implies continuity there.

An analytic function (also called holomorphic or regular functions) in a domain  $D$ , is any function  $f(z)$  which is defined and differentiable at every point in  $D$ . A function is analytic at a point  $z_0$  if it is analytic in some  $\epsilon$  neighborhood ( $|z - z_0| < \epsilon$ ) of  $z_0$ . For example, consider the function  $f(z) = z^n$

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{(z+h)^n - z^n}{h}$$

By the binomial theorem

$$f'(z) = (nz^{n-1} + \frac{h}{2}n(n-1)z^{n-2} + \dots + h^{n-1}) = nz^{n-1}.$$

Therefore,  $f'(z)$  exists at every point in the complex plane and is said to be analytic there. (P. 27-30, 6)

Now consider the function  $f(z) = z\bar{z}$ .

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{(z+h)(\bar{z} + \bar{h}) - z\bar{z}}{h}$$

$$= \lim (z + z \frac{\bar{h}}{h} + \bar{h})$$

But  $z = x + iy$ ,  $\bar{z} = x - iy$ ,  $h = \Delta x + i\Delta y$ , and  $\bar{h} = \Delta x - i\Delta y$ , and so

$$f'(z) = \lim_{\Delta y, \Delta x \rightarrow 0} (x - iy) + (x + iy) \frac{(\Delta x - i\Delta y)}{(\Delta x + i\Delta y)} + (\Delta x - i\Delta y)$$

Take any linear path  $y = mx$  ( $m$  a constant real value) in the  $z$  plane. Then

$\Delta y = m\Delta x$ , and

$$\begin{aligned} f'(z) &= \lim (x - imx) + (x + imx) \frac{(\Delta x - im\Delta x)}{(\Delta x + im\Delta y)} + (\Delta x - im\Delta x) \\ &= \lim x(1 - im) + x(1 - im) + \Delta x(1 - im) \\ &= \lim 2x(1 - im) + \Delta x(1 - im) \\ &= 2x(1 - im) \end{aligned}$$

This value, however, depends upon the particular  $m$  chosen. If the point is approached along the line  $y = 0$ , the limit is  $2x$ . If the line is approached along the line  $y = x$ , the limit is equal to  $2x(1 - i)$ . Therefore, provided  $x \neq 0$  the limit does not exist since it is not unique. (P. 539, 5)

Now it can be seen that the complex derivatives do have a feature which the real derivatives do not. To analyze this further, and in order to establish a criterion by which it can be readily determined whether or not a function of a complex variable is differentiable, the Cauchy-Riemann criterion is now developed. Take any complex function  $w = f(z) = u(x, y) + iv(x, y)$ .  $\Delta w = \Delta u + i\Delta v$ , and  $\Delta z = \Delta x + i\Delta y$  and also

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

where  $z_0 = x_0 + iy_0$  and  $\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$  and

$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$ . Now let  $z$  approach  $z_0$  along the real axis ( $\Delta y = 0$ ).

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta x \rightarrow 0} \frac{\Delta u + i\Delta v}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \end{aligned}$$

But letting  $z$  approach  $z_0$  along the imaginary axis ( $\Delta x = 0$ ) it follows that

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta y \rightarrow 0} \frac{\Delta u + i\Delta v}{i\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-i\Delta u}{\Delta y} + \lim_{\Delta y \rightarrow 0} \frac{\Delta v}{\Delta y} \\ &= \frac{Jv}{Jy} - i\frac{Ju}{Jy} \end{aligned}$$

If  $f(z)$  is to be differentiable at  $z_0$  it is necessary that the derivative not depend upon the path along which  $z$  approaches  $z_0$  and therefore by equating real and imaginary parts of the expressions above the following must hold

$$\frac{Ju}{Jx} = \frac{Jv}{Jy} \quad \text{and} \quad \frac{Jv}{Jx} = -\frac{Ju}{Jy}$$

These are called the Cauchy-Riemann equations. These are a necessary condition which must be satisfied if a function of a complex variable is to be differentiable at a point. For example, consider the function  $w = \bar{z} = x - iy$  where  $u(x,y) = x$  and  $v(x,y) = -y$ .

$$\begin{aligned} \frac{Ju}{Jx} &= 1 & \frac{Jv}{Jx} &= 0 \\ \frac{Ju}{Jy} &= 0 & \frac{Jv}{Jy} &= -1 \end{aligned}$$

Therefore, the partial with respect to  $x$  is not equal to the partial of  $v$  with respect to  $y$  at any point in the complex plane and consequently,  $w = \bar{z}$  is nowhere differentiable.

Now it will be shown that under certain conditions the Cauchy-Riemann equations are not only necessary but also sufficient conditions for differentiability at a point. The conditions that must be imposed are that the first partials of  $u$  and  $v$  exist and are continuous in a neighborhood of the point being considered. (In effect, it will now be proven that  $f(z)$  is differentiable in this neighborhood. But since functions which are analytic at an isolated point are of little interest, there is no harm done in imposing these restrictions.) Recall that a function of two real variables is differentiable when the first partials exist and are continuous. Now taking any two points in the neighborhood of the

point, namely  $(x, y)$  and  $(x + \Delta x, y + \Delta y)$ ,  $\Delta u$  can be written as follows:

$$\Delta u = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

and  $\Delta v$  can be written

$$\Delta v = \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y$$

where  $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$  approach 0 as  $\Delta x$  and  $\Delta y$  approach 0.<sup>6</sup> But  $\Delta w = \Delta u + i\Delta v$  and so

$$\Delta w = \left( \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \right) + i \left( \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y \right) + (\epsilon_1 \Delta x + \epsilon_2 \Delta y) + i(\epsilon_3 \Delta x + \epsilon_4 \Delta y)$$

But since the Cauchy-Riemann equations hold by the given conditions

$$\Delta w = \frac{\partial u}{\partial x} (\Delta x + i\Delta y) + i \frac{\partial v}{\partial x} (\Delta x + i\Delta y) + (\epsilon_1 + i\epsilon_3) \Delta x + i(\epsilon_2 + i\epsilon_4) \Delta y$$

According to the definition,  $f'(z) = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta x + i\Delta y}$

Therefore, if it can be shown that as  $\Delta x$  and  $\Delta y$  approach 0 so do the terms

$$(\epsilon_1 + i\epsilon_3) \frac{\Delta x}{\Delta x + i\Delta y} \quad \text{and} \quad (\epsilon_2 + i\epsilon_4) \frac{\Delta y}{\Delta x + i\Delta y}$$

then

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

It can be shown that the above quantities approach 0 as  $\Delta x$  and  $\Delta y$  approach

0 in the following manner:

$$(\Delta x)^2 = (\Delta x)^2$$

$$(\Delta x)^2 \leq (\Delta x)^2 + (\Delta y)^2$$

$$\sqrt{(\Delta x)^2} \leq \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\frac{\sqrt{(\Delta x)^2}}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \leq 1$$

$$|\epsilon_1 + i\epsilon_3| \frac{|\Delta x|}{|\Delta x + i\Delta y|} \leq |\epsilon_1 + i\epsilon_3|$$

and finally

$$\left| (\epsilon_1 + i\epsilon_3) \frac{\Delta x}{\Delta x + i\Delta y} \right| \leq |\epsilon_1 + i\epsilon_3|$$

The same procedure is used for the other term. Therefore, if the first partials

exist and are continuous and if the Cauchy-Riemann equations hold in a region then  $f(z)$  is differentiable there. For example, consider the function  $w = f(z) = e^z$ . It can be proven in the complex field that  $e^{a+b} = e^a e^b$  and therefore

$$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

where

$$u(x,y) = e^x \cos y \quad v(x,y) = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y \quad \frac{\partial v}{\partial y} = e^x \cos y$$

The Cauchy-Riemann conditions are satisfied and therefore the function is differentiable at all points in the complex plane. By the previous discussion

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\ &= e^x (\cos y + i \sin y) \\ &= e^z \end{aligned}$$

The chain rule can also be derived in the complex field. If  $F(z)$  and  $f(z)$  are analytic functions the following relationship is true:

$$\frac{dF(f(z))}{dz} = \frac{dF(f(z))}{df(z)} \frac{df(z)}{dz} \quad (\text{P. 540-2, 5})$$

In this chapter the basic criterion necessary for determining whether or not a function of a complex variable is analytic have been developed. Also some of the rules for complex differentiation have been provided and in some cases where feasible in the context of this work they have been justified. In the next chapter, some of the basic ideas concerning the integration of a complex function will be discussed.

### III. Introduction to Integration

If the reader is unfamiliar with or has forgotten some of the basic ideas concerning line integrals it is advised that he refer himself to Appendix IV before proceeding with this chapter. Also it should be pointed out that the simple closed curves in this chapter are allowed to have corners and that the regions shall be connected and enclosed by simple closed curves. If  $f(z) = u(x,y) + iv(x,y)$ ,  $dz = dx + idy$ , and  $f(z)$  is defined and continuous in some region  $R$  then the integral along some path  $C$  in  $R$  is defined as follows:

$$\begin{aligned}\int_C f(z)dz &= \int_C (u + iv)(dx + idy) \\ &= \int_C (udx - vdy) + i \int_C (vdx + udy)\end{aligned}$$

where the real and imaginary parts of the integral are nothing more than line integrals. If a line integral

$$\int_C M(x,y)dx + N(x,y)dy$$

is to have a value independent of the path chosen in  $R$  then it is necessary and sufficient that  $M$  and  $N$  and their first partials exist and are continuous and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

in  $R$ . Applying these conditions to the line integral of the complex integral above, it follows that if the integral of  $f(z)$  is to be independent of the path of integration in  $R$ , it is necessary and sufficient that  $u$  and  $v$  and their first partials exist and be continuous and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

in the region  $R$ . But these are nothing more than the Cauchy-Riemann equations. In other words, the necessary and sufficient condition for the integral of a complex function,  $f(z)$ , to be independent of the path of inte-

gration in a region  $R$  is that  $f(z)$  be analytic and  $f'(z)$  be continuous in  $R$ . Now if  $C$  is any simple closed curve in the complex plane and  $f(z)$  is analytic on  $C$  and in the region bounded by  $C$  then  $\int_C f(z) dz = 0$ . This last statement is known as Cauchy's Integral Theorem. A stronger form of the theorem can be proven for which  $f(z)$  need only be continuous on the curve and analytic in the region. For the purposes of this paper, however, this need not be done. (P. 543-5, 5)

As an example, consider the function  $f(z) = z^2$ .

$$f(z) = (x + iy)^2 = (x^2 - y^2) + i(2xy)$$

$$u(x, y) = x^2 - y^2 \quad v(x, y) = 2xy$$

Now

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$

Therefore,  $u$ ,  $v$ , and their first partials exist and are continuous everywhere in the  $z$  plane. Consequently, the integral of  $f(z)$  between any two points in the complex  $z$  plane is independent of the path of integration taken between the two points. Now consider the function  $f(z) = \bar{z}$ .

$$f(z) = x - iy$$

$$u(x, y) = x$$

$$v(x, y) = -y$$

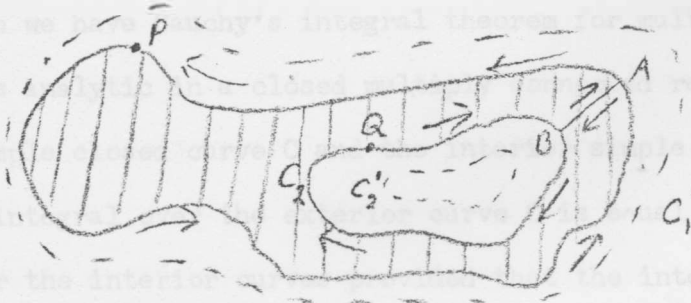
but

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$$

and therefore, there is no region in the  $z$  plane such that the integral of  $f(z)$  between any two points in that region does not depend upon the path taken between the two points.

The next purpose of this chapter shall be to extend Cauchy's Integral Theorem from simply connected regions to multiply connected regions. Take

a multiply connected region bounded by the curves  $C_1$  and  $C_2$  (see figure below). This is a doubly connected region but the ideas presented can readily be extended to an  $n$ -tuply connected region. Assume that  $f(z)$  is a function analytic in the region bounded by the curves  $C_1$  and  $C_2$ . This implies that  $f(z)$  is



analytic on  $C_1$  and  $C_2$ . Take a line connecting a point  $A$  on  $C_1$  with a point  $B$  on  $C_2$  and cut the region along this line to get a simply connected region to which Cauchy's Integral Theorem can be applied. Doing so we get the following:

$$\oint_{APA} f(z) dz + \int_{AB} f(z) dz + \oint_{BQB} f(z) dz + \int_{BA} f(z) dz = 0$$

where the arrows and subscripts indicate the direction of integration. But it is also known that

$$\int_{AB} f(z) dz + \int_{BA} f(z) dz = 0$$

and therefore that

$$\oint_{ABA} f(z) dz + \oint_{BQB} f(z) dz = 0$$

or

$$\oint_{ABA} f(z) dz - \oint_{BQB} f(z) dz = 0$$

and then

$$\oint_{ABA} f(z) dz = \oint_{BQB} f(z) dz.$$

Note, however, that even though the integrals over the two paths are equal, they need not equal 0 since  $f(z)$  does not have to be analytic at every point bounded by  $C_2$ . What has been shown, in effect, is that the integral of any

analytic function over a simple closed curve is equal to the integral of that function over any simple closed curve which can be obtained by a continuous deformation of the original curve without passing any singular points (points at which the function ceases to be analytic) in this deformation. By generalizing then we have Cauchy's integral theorem for multiply connected regions: If  $f(z)$  is analytic in a closed multiply connected region bounded by the exterior simple closed curve  $C$  and the interior simple closed curves  $C_1, C_2, \dots, C_n$  then the integral over the exterior curve  $C$  is equal to the sum of the integrals over the interior curves provided that the integration over all curves (contours) is in the same direction. (P. 546-7, 5)

Now consider a function  $f(z)$  such that  $|f(z)| \leq M$  on a curve  $C$ . Recalling the fact that complex numbers can be represented as two dimensional vectors in the complex plane and that the triangular inequality holds for the magnitudes of vectors, it follows that for  $n$  vectors

$$\left| \sum_{i=1}^n z_i \right| \leq \sum_{i=1}^n |z_i|$$

Knowing that an integral is nothing more than the limit of a sum, it can be shown that

$$\left| \int_C f(z) dz \right| \leq \int_C |f(z) dz|$$

But  $|f(z)| \leq M$  for all  $z$  on  $C$  and therefore

$$\left| \int_C f(z) dz \right| \leq M \int_C |dz| = \int_C \sqrt{dx^2 + dy^2} = \int_C ds$$

The integral on the right then is nothing more than the arc length  $L$  of  $C$  and therefore

$$\left| \int_C f(z) dz \right| \leq ML$$

The fundamental Theorem of the Integral Calculus can now be proven. Require that  $f(z)$  be analytic in a simply connected region  $R$  and pick two points  $z_0$  and  $z$  in that region and consider the integral

$$\int_{z_0}^z f(z) dz$$

Since  $f(z)$  is analytic in  $R$ , this integral depends entirely upon the points  $z_0$  and  $z$  which are chosen, and not upon the path of integration. If  $z_0$  is a fixed value, then the integral can be considered to be a function of  $z$ , that is

$$F(z) = \int_{z_0}^z f(\zeta) d\zeta$$

where the variable  $\zeta$  is used to emphasize the difference between the integration variable and the variable bound on the integral. The next consideration will be to show that  $F'(z) = f(z)$  for all  $z$  in  $R$ . Using the definition of  $F(z)$  above, the quotient-difference can be written

$$\begin{aligned} \frac{F(z + \Delta z) - F(z)}{\Delta z} &= \frac{1}{\Delta z} \left( \int_{z_0}^{z+\Delta z} f(\zeta) d\zeta - \int_{z_0}^z f(\zeta) d\zeta \right) \\ &= \frac{1}{\Delta z} \left( \int_{z_0}^z f(\zeta) d\zeta + \int_z^{z+\Delta z} f(\zeta) d\zeta - \int_{z_0}^z f(\zeta) d\zeta \right) \\ &= \frac{1}{\Delta z} \left( \int_z^{z+\Delta z} f(\zeta) d\zeta \right) \\ &= \frac{1}{\Delta z} \int_z^{z+\Delta z} (f(z) + f(\zeta) - f(z)) d\zeta \\ &= \frac{1}{\Delta z} (f(z) \int_z^{z+\Delta z} d\zeta) + \frac{1}{\Delta z} \left( \int_z^{z+\Delta z} (f(\zeta) - f(z)) d\zeta \right) \end{aligned}$$

The last step is possible since  $z$  is any arbitrary but fixed value in  $R$ . But

$$\int_z^{z+\Delta z} d\zeta = \int_x^{x+\Delta x} dx + i \int_y^{y+\Delta y} dy = \Delta x + i\Delta y = \Delta z$$

and therefore

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z) + \frac{1}{\Delta z} \left( \int_z^{z+\Delta z} (f(\zeta) - f(z)) d\zeta \right)$$

If it can be shown that as  $\Delta z$  approaches 0 so does the last term on the right then it has been proven that  $F'(z) = f(z)$ . If  $M = \max |f(\zeta) - f(z)|$  on the path between  $z$  and  $z + \Delta z$ , then

$$\left| \frac{1}{\Delta z} \int_z^{z+\Delta z} (f(\zeta) - f(z)) d\zeta \right| \leq M$$

The existence of  $M$  is assured by the fact that  $f(\zeta) - f(z)$  is a continuous

function on a compact (closed and bounded) set. Recall that in the real field if a function of one variable is continuous on a closed interval then the function obtains a maximum on that interval. Similarly, it can be proven that if a function of two variables is continuous on a compact set then it reaches a maximum there.<sup>5</sup> Now if  $f(\zeta)$  is continuous on a compact set then its component functions  $u$  and  $v$  must be continuous there. This follows directly from the definition of continuity,  $f(z)$  is constant and therefore continuous. Now since  $f(\zeta) - f(z)$  is continuous on the set and so are its component functions. The magnitude function is then a continuous function of the continuous functions  $u^2$  and  $v^2$  and consequently, it too is continuous on the set. It therefore reaches a maximum  $M$  on the compact set, - the segment from  $z$  to  $z + \Delta z$ . Finally, as  $z$  approaches  $0$ ,  $\zeta$  will have to approach  $z$  since  $\zeta$  is always on the segment from  $z$  to  $z + \Delta z$ , the path of integration. But if  $\zeta$  approaches  $z$  then  $M$  will have to approach  $0$  and since

$$\left| \frac{1}{\Delta z} \int_z^{z+\Delta z} (f(\zeta) - f(z)) d\zeta \right| \leq M$$

it follows that  $F'(z) = f(z)$  in  $R$ .

The next thing to be proven is that if  $F(z)$  is called a primitive of  $f(z)$  then any two primitives of a given function can differ at most by a constant. Call the two primitives  $F(z)$  and  $G(z)$ . Since

$$F'(z) = G'(z) = f(z)$$

by the definition of a primitive it follows that

$$F'(z) - G'(z) = \frac{d}{dz} (F(z) - G(z)) = \frac{dH(z)}{dz} = 0$$

where  $H(z) = F(z) - G(z)$ . But

$$\frac{dH}{dz} = \frac{ju}{jx} + i \frac{jv}{jx} = \frac{ju}{jy} + i \frac{jv}{jy} = 0$$

or

<sup>5</sup>For reference see Olmsted, John M. H. Real Variables. New York: Meredith Corporation, 1959. See pages 321-2.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0$$

and therefore,  $u$  and  $v$  do not depend upon  $x$  and  $y$ . We have then that

$$\int_{z_0}^z f(z) dz = F(z) + C$$

To evaluate  $C$ , let  $z = z_0$ .

$$\int_{z_0}^{z_0} f(z) dz = F(z_0) + C = 0$$

and so

$$C = -F(z_0).$$

From this it follows that

$$\int_{z_0}^z f(z) dz = F(z) - F(z_0)$$

This is called the Fundamental Theorem of the Integral Calculus for functions which are analytic in some region.

Consider, for example, the function

$$f(z) = 1/(z-a)$$

where  $a$  is some constant. It can easily be shown that  $f(z)$  is analytic in the complex plane except at the point  $z = a$ . Now consider

$$\int_C \frac{dz}{z-a}$$

where  $C$  is a closed path. If the point  $a$  lies on  $C$  the integral is improper and requires special considerations just as do their counter parts in the real field. If, on the other hand,  $a$  is in the interior region bounded by  $C$ , then the integral can be evaluated. By circumscribing a circle  $\gamma$  of radius  $r$  about the point  $a$ , a multiply connected region is formed whose interior boundary is the curve  $\gamma$ . Since  $f(z)$  is analytic in the closed region bounded on the exterior by  $C$  and on the interior by  $\gamma$ , Cauchy's Integral Theorem can be applied.

$$\int_C \frac{dz}{z-a} = \int_\gamma \frac{dz}{z-a}$$

Noting that  $z = a + re^{i\theta}$ , where  $0 \leq \theta < 2\pi$  and  $r$  is constant, describes the circular path  $\gamma$  about  $a$ , the substitution  $z - a = re^{i\theta}$  and  $dz = ire^{i\theta}d\theta$  can be used to yield

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = \int_0^{2\pi} id\theta$$

Now using the fundamental theorem it follows that

$$\int_C \frac{dz}{z-a} = 2\pi i$$

whenever  $a$  is in the region bounded by  $C$ . (P. 546-50, 5)

The next and last thing that will be derived in this paper is a rather remarkable relation known as Cauchy's Integral Formula. The formula says, in effect, that given any function  $f(z)$  which is continuous and analytic in a region  $R$  and continuous on the curve  $C$  bounding  $R$ , and if the values of  $f(z)$  are known for all  $z$  on  $C$ , then the value of the function  $f(a)$  for any  $a$  in the interior of  $R$  can be determined. To begin, consider the function

$$1.) \quad f(z)/(z-a)$$

which is a continuous function on  $R + C$ , the closed region, and which is analytic in  $R$  with the possible exception at the point  $z = a$  for both. Exclude the point  $z = a$  from the region by circumscribing a circle  $\gamma$  of radius  $r$  about the point  $a$ , and interior to the curve  $C$ . The function, 1., will then be analytic in the region exterior to  $\gamma$  and interior to  $C$ . By Cauchy's integral theorem it follows then that

$$\int_C \frac{f(z)dz}{z-a} = \int_\gamma \frac{f(z)dz}{z-a}$$

where the direction of integration must be the same. Rewrite the right hand integral as follows:

$$\int_{\gamma} \frac{f(z) dz}{z-a} = \int_{\gamma} \frac{f(z) - f(a)}{z-a} dz + f(a) \int_{\gamma} \frac{dz}{z-a}$$

However,

$$\int_{\gamma} \frac{dz}{z-a} = 2\pi i$$

as was shown before, and therefore

$$\int_{\gamma} \frac{f(z)}{z-a} dz = \int_{\gamma} \frac{f(z) - f(a)}{z-a} dz + f(a) 2\pi i$$

But it can be shown by an analysis similar to that used in proving the fundamental theorem that since  $f(z)$  is continuous

$$\int_{\gamma} \frac{f(z) - f(a)}{z-a} dz = 0$$

Consequently,

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) dz}{z-a}$$

where  $a$  is any point in the interior of  $R$ . The formula is more commonly written

$$2.) \quad f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z}$$

This is known as Cauchy's Integral Formula. As is the case with Cauchy's integral theorem, this formula can also be extended and proved for any multiply connected region.

With the aid of this formula, it is now a simple matter to show that if a function is analytic then it has derivatives of all orders. Now if  $f(z)$  is continuous on a simple closed curve  $C$ , then the integral, 2., defines an analytic function, call it  $F(z)$ , on the interior of  $C$ . This can be shown in the following manner:

$$\begin{aligned} F'(z) &= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z - \Delta z} - \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z} \right) \\ &= \lim_{\Delta z \rightarrow 0} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z - \Delta z)(\zeta - z)} \right) \end{aligned}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^2}$$

And similarly

$$F''(z) = \frac{2!}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^3}$$

.....

$$F^n(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}}$$

Therefore, the derivatives of all orders exist. (P. 551-3, 5)

## Appendix I. Regions

Any finite or infinite collection of complex numbers which are selected according to a given rule is a set of complex numbers. The individual numbers (Unless otherwise stated, the word number shall always refer to a complex number.) are called elements or members of the set, and a set is well-defined if it can be determined whether any given number is or is not a member of the set. If the set contains no members, it is said to be empty.

A set  $S$  of complex numbers is said to be bounded if for all  $z$  in  $S$

$$|z| \leq K \quad \text{where } z = x + iy \text{ and } |z| = \sqrt{x^2 + y^2}$$

$K$  is called the bound for the set  $S$ . For example, the set

$$T = \{z: |\operatorname{Re}(z)| \leq h \text{ and } |\operatorname{Im}(z)| \leq k\}$$

is bounded since  $|z| \leq \sqrt{h^2 + k^2}$  and  $\sqrt{h^2 + k^2}$  is the bound for the set.

The set of numbers which are not members of a given set is called the complement of that given set. The complement of the set above is the set

$$T' = \{z: |\operatorname{Re}(z)| > h \text{ or } |\operatorname{Im}(z)| > k\}$$

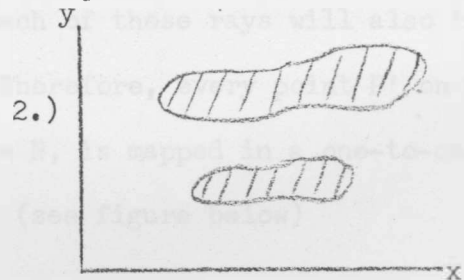
$T'$  is not bounded.

If  $\zeta$  is a point in the complex plane (Because of the one-to-one correspondence between the elements of the set of all complex numbers and the points in the complex plane, the terms number and point can be interchanged.) such that for every  $\epsilon$ -neighborhood of  $\zeta$  there are an infinite number of points of a given set in that neighborhood, then  $\zeta$  is called a limit point of the set. In the sets  $T$  and  $T'$ , every element is a limit point. In the set  $\{x + iy: x \text{ and } y \text{ are integers}\}$  however, there exist no limit points.

An interior point of a set is a point such that there exists an  $\epsilon$ -neighborhood of the point which belongs entirely to the set. For the set  $T$ , any  $z$

an element of  $\{z: |\operatorname{Re}(z)| < h \text{ and } |\operatorname{Im}(z)| < k\}$  is an interior point of  $T$ . Every element of the set  $T'$  is an interior point of  $T'$ . This type of a set, where all elements are interior points, is called an open set. If a set contains all of its limit points, it is said to be a closed set. For example,  $\{x + iy: x \text{ and } y \text{ are integers}\}$  is closed. For a proof assume that the set does not contain all of its limit points. This implies that there exist limit points for the set, which is a contradiction. Therefore, the set must contain all of its limit points, and is therefore closed.

A connected point set is a set such that any two points of the set can be joined by a segmented arc which belongs entirely to the set. For example, the set in figure 1 is a connected point set, while the set in figure 2 is not. A region is a non-empty, open, and connected point set. The set  $T'$  is



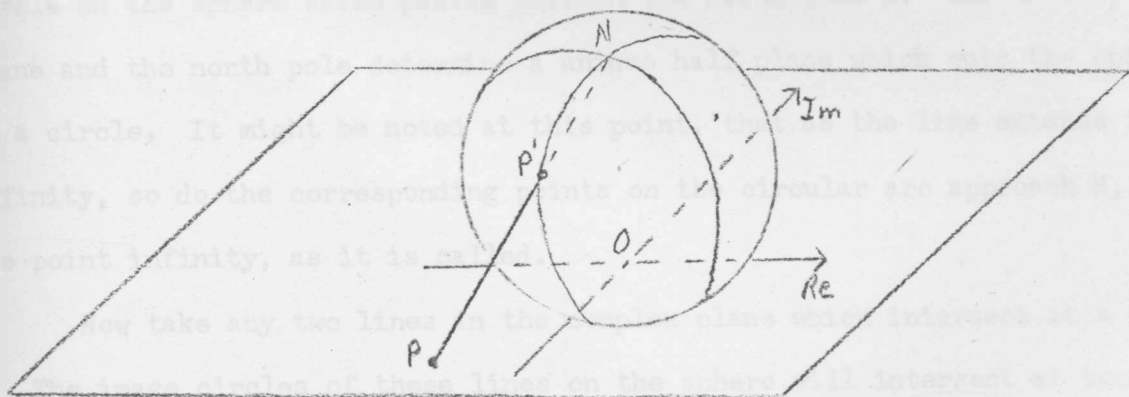
a region, while the set  $T$  is not. (P. 66-8, 3) A region is simply connected if every simple closed curve in the region can be shrunk by continuous deformation to a point on the interior of the curve without crossing the boundaries of the region. A region which is not simply connected is said to be multiply connected. The region in figure 1 is multiply connected. The two regions in figure 2 are each simply connected. (P. 368, 5)

The stereographic projection has two important properties. First of all, the mapping is circular, that is to say that every circle in the sphere maps into a circle or a straight line in the complex plane. The converse of the previous statement is also true. Secondly, the mapping is bijective, which

## Appendix II. The Stereographic Projection

In many instances in the study of the functions of a complex variable it becomes advantageous, if not necessary, to analyze the geometry of the functions on the surface of the sphere rather than on the complex plane. For this purpose, a one-to-one mapping between the points of the complex plane and the points on the surface of the Riemann sphere shall be established in this appendix. This mapping is called the stereographic projection. It is established in the following manner:

A sphere of diameter one is placed on the complex plane at the origin. The point of contact is called the south pole  $O$ , and the opposite point is called the north pole  $N$ . Issuing from the north pole are rays which intersect the sphere at points different from  $N$ . Each of these rays will also intersect the complex plane at a unique point. Therefore, every point  $P'$  on the surface of the sphere, excluding the north pole  $N$ , is mapped in a one-to-one manner onto the points  $P$  of the complex plane. (see figure below)



The stereographic projection has two important properties: First of all, the mapping is circular, that is to say that every circle on the sphere maps into a circle or a straight line on the complex plane. The converse of the previous statement is also true. Secondly, the mapping is isogonal, which

means that any two curves on the sphere intersect at the same angle as do their images in the plane, and conversely. Following is a proof of the above properties.

Definitions:

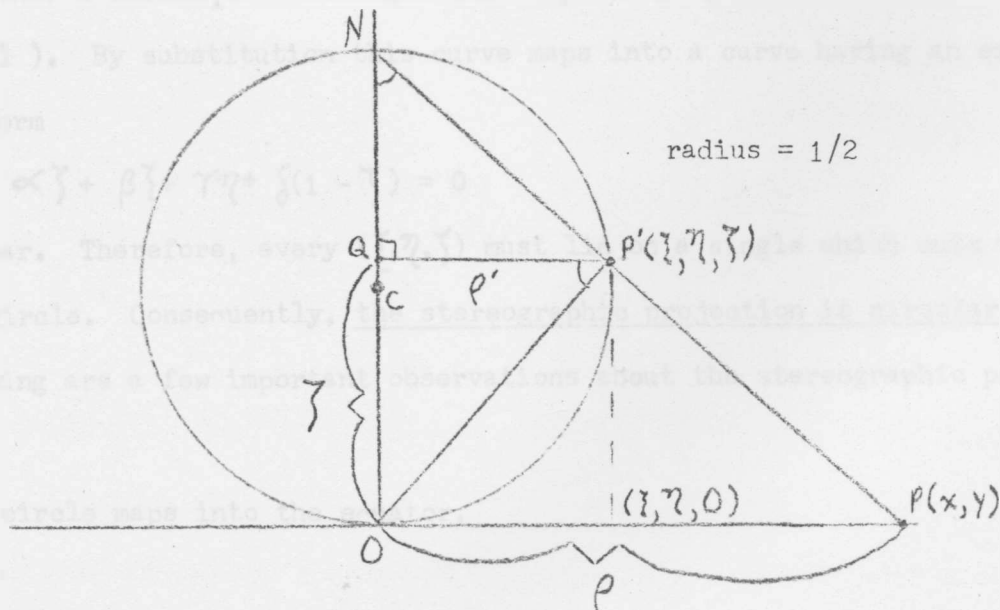
1. Parallels of latitude- Those circles on the sphere which map into the concentric circles in the plane about the point  $O$ . (The equator maps into the unit circle.)
2. Semimeridians- The curves on the sphere which map into the rays in the plane which emanate from  $O$ .
3. Prime meridian- that semimeridian which corresponds to the positive real axis.
4. Geographical longitude- (of a definite semimeridian) is that angle which the semimeridian's corresponding ray in the plane makes with the positive real axis. Counterclockwise angles are positive.

It is easy to show that any line in the complex plane maps into a circle on the sphere which passes through the north pole  $N$ . The line in the plane and the north pole determine a unique half plane which cuts the sphere in a circle. It might be noted at this point, that as the line extends to infinity, so do the corresponding points on the circular arc approach  $N$ , or the point infinity, as it is called.

Now take any two lines in the complex plane which intersect at a point  $P$ . The image circles of these lines on the sphere will intersect at two points, namely  $P'$  which is the image of  $P$ , and the point  $N$ , since all line images pass through this point (infinity). Recall that if two circles intersect in two distinct points, then the angles of intersection are equal. Therefore, in order to show that the mapping of lines in the plane to circles on

the sphere is isogonal it will be sufficient to show that the lines tangent to the circles at N intersect at the same angles as do their image lines in the plane. Note that the line tangent to the circle is parallel to the line in the complex plane. Therefore, the tangent lines at N intersect at the same angles as do the lines in the complex plane. To generalize this, take any two curves  $C_1$  and  $C_2$  which intersect in the complex plane at a point P. These will map into the curves  $C'_1$  and  $C'_2$  on the sphere which will intersect in at least one point P' which is the image of P. The curves  $C_1$  and  $C_2$  will have tangent lines at the point P which will map into circles tangent to  $C'_1$  and  $C'_2$  on the sphere. But, as was shown before, this mapping is isogonal and, therefore, the generalized mapping is isogonal. It can be said then that the stereographic projection is an isogonal mapping.

In order to establish the circularity of the mapping it will be necessary to analyze the projection in three space. To this end a set of  $\xi, \eta, \zeta$  axes are superimposed such that the  $\xi$ -, and  $\eta$ - axes correspond to the x and y (real and imaginary) axes in the plane, and the  $\zeta$ -axis is in the direction of the ON diameter of the sphere. Now consider the plane determined by any point P in the complex plane (other than 0), its image P', and the point O. (see figure below)



From this the following geometric relationships can be seen:

$$\zeta : \rho' = \rho : 1 \quad \text{and} \quad \rho' : (1 - \zeta) = \rho : 1$$

and consequently

$$\rho' = \frac{\rho}{1 + \rho^2} \quad \text{and} \quad \zeta = \frac{\rho^2}{1 + \rho^2}$$

Let  $\theta$  be the angle which the ray OP makes with the positive real axis, then

$$\rho^2 = x^2 + y^2$$

$$x = \rho \cos \theta$$

$$x' = \rho' \cos \theta$$

$$y = \rho \sin \theta$$

$$y' = \rho' \sin \theta$$

Combining these with the expressions above, the following is immediate:

$$\zeta = \frac{x^2}{1 + x^2 + y^2} \quad \eta = \frac{y^2}{1 + x^2 + y^2} \quad \zeta = \frac{x^2 + y^2}{1 + x^2 + y^2}$$

and then

$$x = \frac{\zeta}{1 - \zeta}$$

$$y = \frac{\eta}{1 - \zeta}$$

$$\rho^2 = \frac{\zeta}{1 - \zeta}$$

Note that there is one restriction, and that is that  $\zeta \neq 1$  or, that is to say, that  $(\zeta, \eta, \zeta) \neq N$ .

Now to the circularity of the mapping is a simple matter. The equation

$$\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0$$

represents either a circle or a straight line depending upon whether  $\alpha$  does or does not equal 0. By substitution this curve maps into a curve having an equation of the form

$$\alpha \zeta + \beta \zeta + \gamma \eta + \delta(1 - \zeta) = 0$$

which is linear. Therefore, every  $(\zeta, \eta, \zeta)$  must lie on a single which cuts the sphere in a circle. Consequently, the stereographic projection is circular.

Following are a few important observations about the stereographic projection:

1. The unit circle maps into the equator.

2. The interior of the unit circle maps into the southern hemisphere, while the exterior goes over into the northern hemisphere.
3. The circles in the plane about 0 as center correspond to the parallels of latitude on the sphere. Corresponding to any of these circles in the plane with radius  $r$  is the parallel of latitude having a geographical latitude of  $\beta$ , where

$$\cot(\pi/4 - \beta/2) = r$$

4. A reflection with respect to the axis of reals in the complex plane corresponds to a reflection in the plane of the prime meridian on the complex sphere. (P. 35-40, 3)

Appendix III. DeMoivre's Formulas

If  $u = |u|(\cos\theta + i\sin\theta)$  and  $w = |w|(\cos\phi + i\sin\phi)$  are any complex numbers where  $|u|$  and  $|w|$  are the magnitudes, and  $\theta$  and  $\phi$  are the amplitudes of  $u$  and  $v$  respectively, to get the product of  $u$  and  $v$  multiply the above polar equivalents:

$$\begin{aligned} uw &= (|u|(\cos\theta + i\sin\theta))(|w|(\cos\phi + i\sin\phi)) \\ &= |u||w|(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= |u||w|(\cos\theta\cos\phi + i\sin\theta\cos\phi + i\sin\phi\cos\theta - \sin\phi\sin\theta) \\ &= |u||w|((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \sin\phi\cos\theta)) \end{aligned}$$

Using the basic trigonometric identities

$$1.) \quad uw = |u||w|(\cos(\theta + \phi) + i\sin(\theta + \phi))$$

Now given any complex number  $u = |u|(\cos\theta + i\sin\theta)$  it follows that

$$2.) \quad u^{-1} = (1/|u|)(\cos(-\theta) + i\sin(-\theta))$$

since  $uu^{-1} = 1$  by equation one. (P. 294-5, 6) Using mathematical induction, it can now be shown that given any  $u = |u|(\cos\theta + i\sin\theta)$

$$3.) \quad u^n = |u|^n(\cos n\theta + i\sin n\theta)$$

for any  $n$  which is an element of the set of positive integers. If  $n = 1$

$$u^1 = |u|^1(\cos 1\theta + i\sin 1\theta) = u$$

Assume that the relation holds for  $n = k$ , then

$$u^k = |u|^k(\cos k\theta + i\sin k\theta)$$

and now show that it holds for the  $(k + 1)^{st}$  case under this assumption.

$$u^k u = |u|^k |u|(\cos(k\theta + \theta) + i\sin(k\theta + \theta))$$

by equation one. Simplifying

$$u^{k+1} = |u|^{k+1}(\cos (k+1)\theta + i\sin (k+1)\theta)$$

and therefore, the relationship three holds for all  $n$  an element of the positive integers. Now consider the case where  $n$  is a negative integer.

$$u^n = u^{-(-n)} = (u^{-n})^{-1}$$

where  $-n$  is an element of the positive integers. From equation one it follows then that

$$u^n = (u^{-n})^{-1} = (|u|^{-n}(\cos -n\theta + i \sin -n\theta))^{-1}$$

Now by equation two

$$u^n = |u|^n(\cos n\theta + i \sin n\theta)$$

All that remains for the set of integers is to show that the equation holds true when  $n = 0$ . Since  $0 = n - n$

$$u^0 = u^{n-n} = (u^n)(u^{-n}) = 1 = |u|^0(\cos 0 + i \sin 0)$$

It can be concluded then that given any value  $u = |u|(\cos\theta + i \sin\theta)$ , the following equation

$$u^n = |u|^n(\cos n\theta + i \sin n\theta)$$

is true for all  $n$  when  $n$  is any element of the set of integers.

To extend the theorem so that  $n$  might be any rational number, the following argument can be used. Given any integer  $p$  it would have to be true that if

$$u^{1/p} = |u|^{1/p}(\cos (1/p)\theta + i \sin (1/p)\theta)$$

then

$$(u^{1/p})^p = u$$

According to what has been already established this is the case since

$$\begin{aligned} (u^{1/p})^p &= (|u|^{1/p})^p(\cos(p \cdot 1/p)\theta + i \sin(p \cdot 1/p)\theta) \\ &= |u|(\cos\theta + i \sin\theta) \\ &= u \end{aligned}$$

Therefore, the theorem holds for any  $n$  an element of the set of rational numbers.

It can also be demonstrated that the formula holds true for  $n$ , an element of the reals. Take, for example,  $n = \sqrt{2}$ . It must be true that  $(u^{\sqrt{2}})^{\sqrt{2}} = u^2$ .

$$u^2 = |u|^2(\cos 2\theta + i \sin 2\theta) = (|u|^{\sqrt{2}})^{\sqrt{2}}(\cos \sqrt{2}\sqrt{2}\theta + i \sin \sqrt{2}\sqrt{2}\theta)$$

$$= (\sqrt{2})^2$$

Appendix IV, Line Integrals

Therefore, it would be expected that DeMoivre's theorem is true for any n which is an element of the set of reals.

Let us cut C at n-1 points so as to partition C into n segments. Label the n-1 division points  $(x_1, y_1), (x_2, y_2), \dots, (x_{n-1}, y_{n-1})$  and let  $z_k = x_k + iy_k$ ,  $dz_k = dx_k + iy_k dy_k$  where  $k = 1, 2, \dots, n-1$ , and  $z_n = (x_n, y_n) = (x_0, y_0)$ . Let any point  $(x_k, y_k)$  between the points  $(x_{k-1}, y_{k-1})$  and  $(x_k, y_k)$  and form the sum

$$\sum_{k=1}^n (u(x_k, y_k) dx_k + v(x_k, y_k) dy_k)$$

If the limit as n approaches infinity in such a way that  $z_k$  and  $z_{k+1}$  approach each other then the limit is called the line integral along C. The notation is

$$\int_C (u(x,y) dx + v(x,y) dy) \quad \text{or} \quad \int_{(x_0, y_0)}^{(x_1, y_1)} u dx + v dy$$

It can be proven that if u and v are continuously differentiable at every point on C then the limit does exist. The value of the integral depends upon three things: the particular functions u and v, the end points  $(x_0, y_0)$  and  $(x_1, y_1)$ , and the path of integration C.

There are basically three methods for evaluating line integrals. If the path of integration is given as a function, either  $y = s(x)$  or  $x = g(y)$  then

$$I = \int_{x_0}^{x_1} (u(x, s(x)) dx + s'(x) v(x, s(x)) dy) \quad \text{or} \quad \int_{y_0}^{y_1} (u(g(y), y) dy + g'(y) v(g(y), y) dx)$$

for the first case, and

$$I = \int_{y_0}^{y_1} (u(g(y), y) dx + g'(y) v(g(y), y) dy) \quad \text{or} \quad \int_{x_0}^{x_1} (u(x, g(x)) dx + g'(x) v(x, g(x)) dy)$$

for the second. If the path of integration is given by parametric form,  $x = s(t)$ ,  $y = r(t)$ ,  $(x_1, y_1) = (s(t_1), r(t_1))$  and  $(x_0, y_0) = (s(t_0), r(t_0))$ , then

$$I = \int_{t_0}^{t_1} (u(s(t), r(t)) s'(t) dt + v(s(t), r(t)) r'(t) dt)$$

where  $dx = s'(t) dt$  and  $dy = r'(t) dt$ .

Before proceeding with the proof of Green's Theorem, it should be pointed

## Appendix IV. Line Integrals

Let  $C$  be a curve in the  $xy$  plane with end points  $(a_1, b_1)$  and  $(a_2, b_2)$ . Take  $M(x, y)$  and  $N(x, y)$  to be any single-valued functions which are defined on  $C$ . Now cut  $C$  at  $n-1$  points so as to partition  $C$  into  $n$  segmental arcs. Label the  $n-1$  division points  $(x_1, y_1), (x_2, y_2), \dots, (x_{n-1}, y_{n-1})$  and let  $\Delta x_k = x_k - x_{k-1}$  and  $\Delta y_k = y_k - y_{k-1}$  where  $k = 1, 2, \dots, n$ ,  $(x_0, y_0) = (a_1, b_1)$ , and  $(x_n, y_n) = (a_2, b_2)$ . Pick any point  $(\xi_k, \eta_k)$  between the points  $(x_k, y_k)$  and  $(x_{k-1}, y_{k-1})$  for each  $k$  and form the sum

$$\sum_{k=1}^n (M(\xi_k, \eta_k) \Delta x_k + N(\xi_k, \eta_k) \Delta y_k)$$

If the limit as  $n$  approaches infinity in such a way that  $\Delta x_k$  and  $\Delta y_k$  approach 0 exists then the limit is called the line integral along  $C$ . The notation is

$$\int_C M(x, y) dx + N(x, y) dy \quad \text{or} \quad \int_{(a_1, b_1)}^{(a_2, b_2)} M dx + N dy$$

It can be proven that if  $M$  and  $N$  are sectionally continuous at every point on  $C$  then the limit does exist. The value of the integral depends upon three things: the particular functions  $M$  and  $N$ , the end points  $(a_1, b_1)$  and  $(a_2, b_2)$ , and the path of integration  $C$ .

There are basically three methods for evaluating line integrals. If the path of integration is given as a function, either  $y = f(x)$  or  $x = g(y)$  then

$$I = \int_{a_1}^{a_2} M(x, f(x)) dx + N(x, f(x)) f'(x) dx \quad \text{since } dy = f'(x) dx$$

for the first case, and

$$I = \int_b^{b_2} M(g(y), y) g'(y) dy + N(g(y), y) dy \quad \text{since } dx = g'(y) dy$$

for the second. If the path of integration is given in parametric form,

$x = \Phi(t)$ ,  $y = \Psi(t)$ ,  $(a_1, b_1) = (\Phi(t_1), \Psi(t_1))$  and  $(a_2, b_2) = (\Phi(t_2), \Psi(t_2))$ , then

$$I = \int_{t_1}^{t_2} M(\Phi(t), \Psi(t)) \Phi'(t) dt + N(\Phi(t), \Psi(t)) \Psi'(t) dt$$

where  $dx = \Phi'(t) dt$  and  $dy = \Psi'(t) dt$ .

Before proceeding with the proof of Green's Theorem, it should be pointed

ed out that a simple closed curve is a closed curve which does not intersect itself. Also a simply connected region is a region such that any closed curve in it can be shrunk to a point by continuous deformation without leaving the region. Any other region is called multiply connected. (see also Appendix I)

The familiar properties for integrals can also be proven true for line integrals. They are:

$$\int_C M(x,y)dx + N(x,y)dy = \int_C M(x,y)dx + \int_C N(x,y)dy$$

$$\int_{(a_1, b_1)}^{(a_2, b_2)} Mdx + Ndy = - \int_{(a_2, b_2)}^{(a_1, b_1)} Mdx + Ndy$$

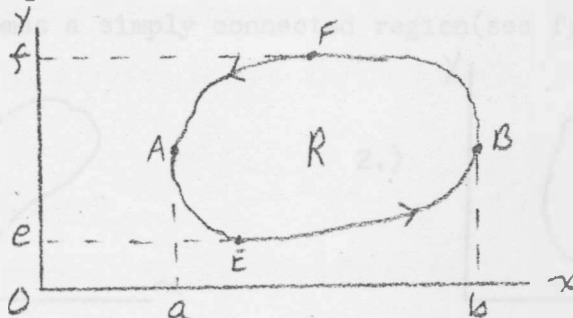
$$\int_{(a_1, b_1)}^{(a_2, b_2)} Mdx + Ndy = \int_{(a_1, b_1)}^{(a_3, b_3)} Mdx + Ndy + \int_{(a_3, b_3)}^{(a_2, b_2)} Mdx + Ndy$$

where  $(a_3, b_3)$  is any point on  $C$  between  $(a_1, b_1)$  and  $(a_2, b_2)$ .

Green's Theorem states that if  $M$ ,  $N$ ,  $\partial M/\partial y$ , and  $\partial N/\partial x$  are single valued and continuous in a simply connected region  $R$  bounded by a simple closed curve  $C$  then

$$\oint_C Mdx + Ndy = \iint_R (\partial N/\partial x - \partial M/\partial y) dx dy$$

The theorem will be proven for a region  $R$  in the  $xy$  plane having the property that any straight line parallel to either of the coordinate axes shall cut the closed curve  $C$ , which is the boundary of the region, in at most two points (see figure below). It can be readily extended and proven for any multiply connected region, although this will not be done here.



The proof is as follows. Let  $y = Y_1(x)$  and  $y = Y_2(x)$  be the equations of the curves  $AEB$  and  $AFB$  respectively (refer to figure above). Then

$$\begin{aligned}
 \iint_R (JM/Jy) dx dy &= \int_a^b \left( \int_{y_1(x)}^{y_2(x)} (JM/Jy) dy \right) dx \\
 &= \int_a^b M(x,y) \Big|_{y_1(x)}^{y_2(x)} dx \\
 &= \int_a^b (M(x, Y_2) - M(x, Y_1)) dx \\
 &= - \int_b^a M(x, Y_1) dx - \int_b^a M(x, Y_2) dx \\
 1.) \quad &= - \oint_C M dx = \iint_R (JM/Jy) dx dy
 \end{aligned}$$

Similarly it can be shown that

$$2.) \quad \iint_R (JN/Jx) dx dy = \oint_C N dy$$

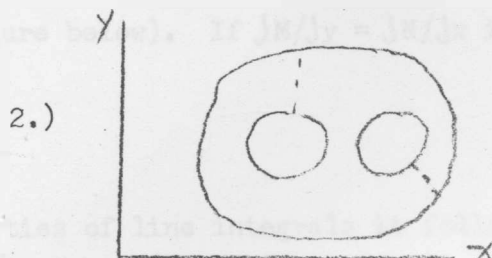
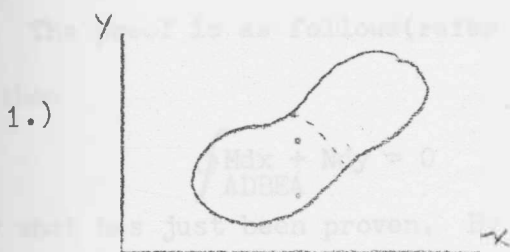
The existence of the integrals

$$\int_{y_1(x)}^{y_2(x)} (JM/Jy) dy \quad \text{and} \quad \int_{x_1(y)}^{x_2(y)} (JN/Jx) dx$$

is insured by the continuity of the two partials. Adding expressions one and two, it follows that

$$\iint_R (JM/Jy - JN/Jx) dx dy = \oint_C M dx + N dy$$

The key that is used in proving the theorem for any simply connected region is in the fact that any such region can be divided into regions which have the property needed in the proof above (see figure 1 below). For multiply connected regions the key is in the fact that any such region can be cut in such a way that it becomes a simply connected region (see figure 2 below).



Now given that  $M(x,y)$  and  $N(x,y)$  are continuous and their first partial derivatives are continuous at every point in a simply connected region  $R$ , then a necessary and sufficient condition for  $\oint_C M dx + N dy = 0$  around every closed path in  $R$  is that  $JM/Jy = JN/Jx$  in  $R$ .

The proof is as follows. If  $\int M/\int y = \int N/\int x$  then by Green's Theorem

$$\oint_C Mdx + Ndy = \iint_R (\int M/\int y - \int N/\int x) dx dy = 0$$

Therefore, the condition is sufficient. To prove the necessity suppose that

$$\oint_C Mdx + Ndy = 0$$

around every closed path in  $R$  and that  $\int M/\int y \neq \int N/\int x$  at some point  $(x_0, y_0)$  in  $R$ . There are then two possibilities:  $\int M/\int y > \int N/\int x$  or  $\int M/\int y < \int N/\int x$ . Take the first case. Since  $\int M/\int y$  and  $\int N/\int x$  are continuous in  $R$  and since  $\int M/\int y - \int N/\int x$  is then continuous and greater than 0 at  $(x_0, y_0)$  it can be proven that there exists some subregion of  $R$ , call it  $R_1$ , about  $(x_0, y_0)$  such that  $\int M/\int y - \int N/\int x > 0$  in  $R_1$ . If  $C_1$  is the boundary of  $R_1$  then by Green's Theorem

$$\oint_{C_1} Mdx + Ndy = \iint_{R_1} (\int M/\int y - \int N/\int x) dx dy > 0$$

which contradicts the original condition that  $\oint_C Mdx + Ndy = 0$  in  $R$ . A similar argument can be used for the case when  $\int M/\int y < \int N/\int x$ . Consequently, it is necessary that if  $\oint_C Mdx + Ndy = 0$  in  $R$  then  $\int M/\int y = \int N/\int x$  in  $R$ .

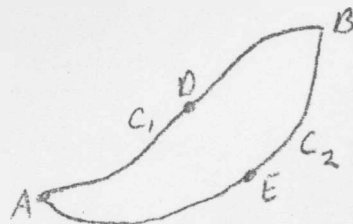
It can now be proven that under these same conditions, that is that  $M$ ,  $N$ , and their first partials exist and are continuous in  $R$ , that  $\int_A^B Mdx + Ndy$  is independent of the path of integration in  $R$ , if and only if  $\int M/\int y = \int N/\int x$  in  $R$ . The proof is as follows (refer to the figure below). If  $\int M/\int y = \int N/\int x$  in  $R$  then

$$\int_{ADBEA} Mdx + Ndy = 0$$

by what has just been proven. By the properties of line integrals it follows that

$$\int_{ADBEA} = \int_{ADB} + \int_{BEA} = 0$$

$$\int_{ADB} = - \int_{BEA}$$



$$\int_{ADB} = \int_{AEB}$$

Therefore, the condition is sufficient. If the line integral is independent of the path then

$$\int_{C_1} = \int_{C_2} \quad \int_{ADB} = \int_{AEB} \quad \int_{ADBEA} = 0$$

But from the previous proof this implies that  $M/y = N/x$ . (P. 195-7, 7)

1. Lancelotti, Elements of the Theory of Functions, Trans. by Frederick B. Sewell, New York, 1932.
2. George B. Thomas, Jr., Calculus and Analytic Geometry, Reading, Massachusetts, 1960.
3. L. C. Ishakoff and R. M. Ruffner, Mathematical Physics and Modern Engineering, New York, 1964.
4. Louis K. Walther, Introduction to Matrix Algebra, New York, 1958.
5. Murray H. Spiegel, Schaum's Outline Series: The Theory and Problems of Advanced Calculus, New York, 1963.

### List of References

1. Humphrey S. C. Fong, "The Linear Fractional Transformation of a Complex Variable:  $w = (az + b)/(cz + d)$ ." Unpublished honors thesis. Helena, Montana: Department of Mathematics, Carroll College, 1964.
2. E. J. Townsend, Functions of a Complex Variable, New York 1930.
3. Konrad Knopp, Elements of the Theory of Functions, Trans. by Frederick Bagemihl, New York, 1952.
4. George B. Thomas, Jr., Calculus and Analytic Geometry, Reading, Massachusetts: 1968.
5. I. S. Sokolnikoff and R. M. Redheffer, Mathematics of Physics and Modern Engineering, New York, 1966.
6. Louis M. Weiner, Introduction to Modern Algebra, New York, 1970.
7. Murray R. Spiegel, Schaum's Outline Series: The Theory and Problems of Advanced Calculus, New York, 1963.