Mathematics and Electricity

Modeling with an RLC Circuit

By: Joe Havens

Director: Kelly Cline

February 28, 2007

SIGNATURE PAGE

This thesis for honors recognition has been approved for the Department of Mathematics.

	4/2/07
Director	Date
Anthony M. Spilka	4/2/07
Reader	Date
Mark	4.2.07
Reader	Date

Table of Contents

Introduction

Electric circuits can be useful outside of electronics. For example, an electric circuit can be used to model a mechanical mechanism such as a car's suspension. This is possible because like an RLC circuit, a circuit with a resistor, inductor, and a capacitor in series, a suspension system with a mass, a spring, and a shock absorber uses a second order differential equation to model the system.

According to Fundamentals of Physics the equation for a damped mass on a spring is:

Equation 1:

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

The solution for this equation is:

Equation 2:

$$x(t)=x_m * e^{-bt/2m}\cos(\omega' t+\phi)$$

where,

$$\omega'=\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$$

In this equation *b* is the damping coefficient, *m* is the mass, *k* is the spring constant, x(t) is the height of the car as a function of time, and x_m is the amplitude. The exponent is what causes the damping. As *b* increases relative to *m* the function reaches equilibrium quicker.

The equation for the RLC circuit is:

Equation 3:

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=0$$

The solution for this equation is:

Equation 4:

$$q(t)=Q*e^{-Rt/2L}\cos(\omega't+\phi)$$

where,

$$\omega'=\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}.$$

In this equation *R* is the resistance, *L* is the inductance, *C* is the capacitance, q is the voltage across the capacitor, and *Q* is the amplitude. As with the spring model it is the exponent that causes the dissipation (Halliday, Resnick, Walker 2005).

Now that we have the equations we can see the similarities. It is easy to see that *R* is analogous to b, *m* to *L*, *k* to $1/C$, and *x(t)* to *q(t)*. Therefore, if we make the ratios of $\frac{b}{m}$ and $\frac{k}{m}$ equal to $\frac{R}{L}$ and $\frac{1}{LC}$ respectively, we will end up with a x(t) and a q(t) that act the same.

So we can see that this approach can be done. However, the question remains, what is the advantage of using a circuit? The first advantage is that circuit elements are much less expensive than suspension systems. We can also use a circuit to see how a suspension would respond to conditions that are too extreme to drive a car on. The greatest advantage to using circuit elements is that the system can be adjusted with relative ease, compared to replacing springs and shock absorbers. This allows the individual to adjust the system to have the desired response before putting the time and money into making the suspension.

In this thesis paper we will explore the suspension of a 1948 Ford F-2 and attempt to use an RLC circuit to model the suspension. This particular truck was chosen because of its unique and complicated suspension. We will begin by analyzing the truck's suspension mathematically, then we will use the data we get from the truck to design a circuit that will correspond with the suspension.

Collecting Data on the Truck

The first step is to collect the data on the truck. We are especially interested in the spring constant, damping constant, and mass of the truck. Since the most interesting part of the suspension is in the back of the truck, and because it would be too intrusive to this old truck to analyze the front suspension, we will only analyze the rear suspension.

The first, and easiest, thing to gather is the mass of the truck. To gather this information we used the truck scale at a gravel pit. Unfortunately, the scale measures the mass of the truck in increments of 10 kilograms; however, for our purposes this should be accurate enough. We find that the truck has a mass of 1,490 kg. Since we are only interested in the rear suspension, we weigh the truck with only the rear axle on the scale and find that there is 560 kg on the rear axle.

Next we want to find the spring constant of the truck. This is where we first run into some interesting problems due to the uniqueness of the truck. As seen in Figure 1, the truck has two sets of springs: one for normal use and another that gets activated if a large load is placed on it. For the purpose of our model we will assume that the truck has a heavy enough load to activate the overload springs, and that the truck will be going over large enough bumps to lift the load off of the overload springs for a short time. Therefore we must find the spring constant both with and without the overload springs.

The second problem is that the stiffness of the springs is so great that we need a large amount of mass to get them to move. To solve this problem we use a line truck to place reels of overhead wire that weigh about 630 kg each on the back of the truck. In order to get accurate readings on the springs with and without the overload springs we must use three reels of wire.

We begin by placing the rear axle on jack stands to protect the tires, and to ensure that it is only the springs moving. Then we measure the height from the ground to the frame on each side of the truck. We then place the first reel, with a mass of 622 kg, on the truck. When we do this the right side ends up activating the overload springs, because of imperfections in the truck, therefore only the left side, which dropped 0.0317 m, will provide us an idea of the spring constant without the overload springs. By taking half the mass we added to the truck, and multiplying it by 9.81 m/s^2, we get the force that we placed on the truck, which comes out to be 6101.82 N. We then divide this by the 0.0317m drop that occurred and this gives us our spring constant, 96,243.2 N/m, for one side. We then multiply this by two to get the total spring constant for the rear suspension: 192 kN/m.

Next we put another reel of wire, with a mass of 633 kg, on the truck to fully activate both overload springs. Then we add a third reel, with a mass of 653 kg, and repeat the process above to find the spring constant for the rear suspension when the over load springs are activated. This comes out to be 1350 kN/m.

We still need to find the damping constant; however we must do some analysis on suspension before we can attempt to estimate this value. Therefore, we will begin analyzing the suspension with what we have.

Suspension Model (No Damping)

For our first model of the suspension we will look at the suspension with no damping, and we will assume that the rear of the truck moves as one unit, not independently from side to side. Finally we will assume the rear of the truck is at equilibrium, (zero meters,) with 2,468 kg on it, and on the overload springs. Recall from the last section, the spring constant without the overload springs was 192,486.4 N/m, and the spring constant with the overload springs was 1,348,616.84 N/m.

In Fundamentals of Physics we learn that the equation for the motion of an oscillating spring is:

Equation 5:

$$x(t) = x_m \cos(\omega t + \phi),$$

where *x(t)* is the displacement, x_m is the amplitude, ω is the angular frequency, and ϕ is the phase angle. We also learn that ω is found by the equation:

$$\omega = \sqrt{\frac{k}{m}}$$

(Halliday, Resnick, Walker 2005). If we assume an amplitude of .03 m, and a phase shift of $\frac{\pi}{2}$ to shift the function so that $x(0)=0$, we get the function:

$$x(t) = 0.03\cos\left(23.38t + \frac{\pi}{2}\right)$$

when $0<t<0.158$ seconds. This gives us the first part of our overall function. When t = 0.158 seconds $x(t)$ is at 0.016 m which we found from measuring to be the point when the suspension lifts off of the overload springs, therefore we must re-evaluate $x(t)$ with a new spring constant.

Since the velocity of the system doesn't change when the suspension lifts off the overload springs, it is important to ensure that the derivatives of the functions, as well as the functions themselves, are the same at $t = 0.158$ seconds. By finding the derivative and evaluating at $t = 0.158$ seconds we find that $x'(0.158) = 0.597$ m/s. We then use our new ω in our general function for $x(t)$, while leaving the amplitude and the phase angle as unknowns, which gives us the equation:

$$x(t) = x_m \cos(8.83t + \phi) .$$

By solving this new function and its derivative at time $t = 0.158$, so that both the function and the derivative are the same as the first function, we can find values for x_m and ϕ. When we do this we get the next part of our overall function:

$$x(t) = -0.0695\cos(8.83t + 0.40778)$$

when $0.158<t<0.461$ seconds.

At $t = 0.467$ seconds the suspension drops back onto the overload springs. At this point we need to repeat the process above with our original ω. When we do this we find that $x_m = .03$ and $\phi = -3.489$, so the last part of our overall function is:

$$x(t) = 0.03\cos(23.38t - 3.489)$$

when $0.461<t<0.644$ seconds. Now we have a full period, and beyond this point the function has the same behavior as what we have just found.

If we graph our function, as in Figure 2, we can see the difference the spring constant makes in the way the system reacts. Note that there is a shorter period when the overload springs are activated due to the larger spring constant.

Now that we have a model without damping we want to create a model with damping. In order to do this we must estimate our damping coefficient, which we will do next.

Estimate the Damping Coefficient

Unfortunately, we don't have the resources to get a highly accurate value for the damping coefficient, so we have to use some creative means to get the best estimate possible. To do this we will observe how the suspension reacts under a controlled stimulus and then use equation 1.

We will begin by seeing how the rear suspension reacts when we drive the truck over a speed bump. We will observe the time it takes to stop oscillating, along with counting the number of peaks and troughs it has before it stops oscillating. In order to prevent the oscillation of the front suspension affecting the rear suspension we will position the truck with the front tires already over the speed bump, before we start moving the truck.

The speed bump is 0.1 m high so we will assume this is our x_m. In addition the truck has no additional weight on the rear suspension, so we will use a mass of 560 kg, and the spring constant with no overload springs. The truck experiences two troughs and two peaks in about one second's time before stabilizing. By putting this information into the equation above we get:

$$x(t) = 0.1e^{-bt/1120}\left(\cos\sqrt{171.8625 - \frac{b^2}{5{,}017{,}600}}\,t + \frac{\pi}{2}\right)$$

We then graph this with different values for b until we get a graph that has two troughs and two peaks in about one second, before it stops oscillating. When we do this we find that b is about 5,000. The graph of this is given in Figure 3.

Now that we have an estimate of the damping coefficient we are ready to make a model that includes damping in the system.

Suspension Model (With Damping)

We approach the model with damping much the same as we did with the model that didn't include damping. We will begin by analyzing the suspension assuming that it is resting on the overload springs and that it hits a bump that creates an oscillation with a magnitude of 0.03 m. We will still be using the same mass and spring constants, as well as the damping coefficient of 5,000, which we estimated in the last section.

We refer back to equation 1 for an oscillating system with damping found in Fundamentals of Physics:

$$x(t) = x_m * e^{-bt/2m}\cos(\omega' t + \phi)$$

where,

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\,.$$

For the initial part of the model we find that ω'= 23.354, and we make $\phi = \pi/2$ so that $x(0)$=0. This gives us the function:

$$x(t) = 0.03 * e^{-5000*t/4936}\cos\left(23.354*t + \frac{\pi}{2}\right)$$

when 0<t<0.164 seconds. At 0.164 seconds the suspension lifts off of the overload springs at a height of 0.016 m, and a velocity of 0.445 m/s.

Now we have to find the next portion of the function by finding an amplitude and phase shift so that $x(t)$=.016 and $x'(t)$=0.445. When we use the smaller spring constant, we end up with an ω' of 8.773. We then set the function and the derivative equal to 0.016 and 0.445 respectively, and solve for x_m and ϕ. This gives us an x_m of -0.0646 and a ϕ of 0.423. So the next part of our function is:

$$x(t) = -.0646 * e^{-5000t/4936}\cos(8.773t + .4228)$$

when 0.164<t<0.444 seconds. At 0.444 seconds the suspension gets back on the overload springs at a velocity of -0.445 m/s.

We repeat the procedure above with our original ω' to find the last part of our function. When we do this we find x_m=-0.0336 and ϕ=–6.488. So the third part of the equation is:

$$x(t) = 0.0336 * e^{-5000*t/4936}\cos(23.354*t - 6.488)$$

when 0.444<t<0.668 seconds. However, at 0.668 seconds it rises off of the overload springs 0.008 m for about 0.025 seconds, which is negligible, and for our purposes it isn't worth analyzing the effect this period of time has on the model. Therefore, we will assume that this last part of the equation works from 0.444 seconds to ∞. When we graph the entire function, we get the graph shown in Figure 4.

Now that we have created a model with damping we are almost ready to build a circuit that will act in a similar matter. First, however, we must decide what circuit elements to use.

Building the Circuit

As promised we will construct an RLC circuit, which is a circuit with a resistor, an inductor, and a capacitor in series. We will measure the voltage across the capacitor as a function of time. This should give us a wave form similar to that we obtained when we graphed the movement of the suspension as a function of time.

For our first circuit we will make a model that simulates the suspension with no overload springs. As we discussed in the introduction, it would be ideal to choose our circuit elements so the ratios of $\frac{b}{m}$ and $\frac{k}{m}$ are equal to $\frac{R}{L}$ and $\frac{1}{LC}$ respectively. However, $\frac{b}{m}$ equals about 2.02, so R would have to be just over twice the size of L. The problem we run into with doing this is that the largest inductors we have are 50 mH, and we have a resistance of 50 ohms from the power source alone. Therefore it isn't reasonable to satisfy these ratios.

Because of this we will turn to making the damping ratios, ζ, of the two systems the same. In The Shock Absorber Handbook by John Dixon we learn the equation for ζ is:

$$\zeta = \frac{\alpha}{\omega}$$

where,

$$\alpha = \frac{b}{2m} \text{ and } \omega = \sqrt{\frac{K}{m}}$$

(Dixon 1999).

Similarly in Electrical Engineering by Giorgio Rizzoni we are given the same equation for ζ where,

$$\alpha = \frac{R}{2L} \text{ and } \omega = \frac{1}{\sqrt{LC}}$$

(Rizzoni 2007).

So we simply need to come up with circuit elements that give us the same ζ as the suspension.

When we solve for α and ω in the suspension, with the overload springs, we get 1.013 and 8.83 respectively. This gives us a ζ of 0.115. So we need to find some combination of circuit elements that will give us the same ζ. We will choose the inductor first, because of our limited number of inductors, then the capacitor, and finally the resistor, keeping in mind the resistance of the power source and the inductors. The result is the circuit shown in Figure 5.

This circuit gives us a ζ of 0.122, which gives us a percent error of 6.37% from the ζ of the suspension. When we measure the voltage across the capacitor we get the graph shown in Figure 6. In Figure 6 the input voltage, which is shown by the flat line, is a square-wave with the transition occurring at the very beginning of the window. The oscillating line is the voltage across the capacitor.

As you can see there is a great deal of noise in the reading that is preventing us from getting an accurate picture. Additionally all the oscillation has nearly stopped after only about 3 μs. This is because of a very small time constant which is brought about because of the small capacitance. Therefore, when we do our next model we will use a larger capacitor.

A Final Circuit

Now we will make a circuit that will not only model the suspension without the over load springs but will also transition to a different ζ, at a given voltage. First we will determine what values of ζ we need. Secondly, we will look at the various components that will allow us to make the transition. Finally, we will choose the circuit elements and build the circuit.

As we found earlier, $\zeta=0.115$ when the overload springs are not activated. When the overload springs are activated our α remains the same, however, our ω becomes 23.376. This gives us a ζ of 0.0433.

Now the question becomes how we can change the value of ζ in our circuit when the voltage across the capacitor reaches a certain value. In order to do this we will use a 311 op Amp, and a MC14016B switch. For pin assignments on these components see Appendix A. The switch is closed when the control voltage is the same as V_{DD}—5 volts in our case. The switch opens when the control voltage is the same as V_{ss}—0 volts in our case.

The control voltage will come from the op Amp which will produce an output voltage of either 5 or 0 volts. It will change depending on V_C, the capacitor voltage, relative to V_T, the trigger voltage. When $V_C < V_T$, the op Amp will put out 5 volts, and the switch will close; when $V_C > V_T$, the op Amp will put out 0 volts, and the switch will open.

However, the MC14016B does have a resistance associated with it. Therefore, before we can use it we must know what the resistance is. In order to measure the resistance across the switch we will build the circuit shown in Figure 7. We measure R in the circuit, which turns out to be 985Ω. We then measure the voltage across the resistor, which turns out to be 1.85 volts, and divide it by the voltage we put in, 2.5V. We also put 5 volts into the control voltage and make V_{DD} 5 volts so the switch will be closed. Then we are left with a simple voltage divider and we simply solve for R_{SW}, the resistance of the switch, in the equation below:

$$\frac{1.85V}{2.5V} = \frac{985\Omega}{985\Omega + R_{SW}}.$$

The result is R_{SW}=346Ω.

Now we are finally ready to build our final circuit. We begin by making a simple RLC circuit that gives us a ζ of 0.115. We again choose the inductors first, then the capacitor, and finally the resistor. When we do this we end up using four inductors, each of which is about 48 mH and has a resistance of 30Ω. We also use a capacitor of

0.008μF. When we choose our resistor we must remember to subtract the 50Ω of the voltage source and the 120Ω of the inductors. We end up using a 958Ω resistor. This gives us a ζ of 0.115, which is exactly what we wanted.

Next we put the switch in parallel with the resistor along with another resistor. This will cause the total resistance to drop when the switch is closed. Since the only thing we are changing is R, only α will change when we go to get our new ζ. As mentioned above, we are looking for a ζ of 0.0443. We will use a 5.12Ω resistor in series with the switch to give us a ζ of 0.0436 which gives us a percent error of 1.63%.

Now we need to set V_T. We do this by using a voltage divider between 5V and ground and put the voltage across the resistor that goes to ground into V_+ on the op Amp. We will put V_C into V_- on the op Amp. Recall that the suspension transitioned off of the overload springs at 0.016 m and that we use an amplitude of .03 with the suspension. We will be using an amplitude of 2V peak-to-peak as our input voltage, and we will use a dc offset of 2V, in order to protect our switch, therefore we can determine the voltage at which we want the transition by using the equation:

$$\frac{1V}{V_t} = \frac{.03m}{.016m}.$$

We then add 3 to V_t to get V_T. The result is V_T=3.533V.

Now we hook everything up and get the circuit shown in Figure 8. The result of this circuit is shown in Figure 9A. Figure 9B shows the result of the circuit with the switch always closed, as if the suspension never left the overload springs. Figure 9C shows the result if the switch were always open, as if the suspension was never on the overload springs.

According to The Shock Absorber Handbook, the damping ratio tells us how long it will take the system to return to equilibrium (Dixon 1999). Therefore, we want to compare the time it takes for the circuit to reach equilibrium compared to the suspension. Because the circuit was designed to have the same damping ratio as the suspension, we expect it to take the same amount of time relative to its amplitude. When we compare the graphs in Figure 9 with the graph in Figure 4 we see that each part of the system does dissipate at a proportional rate.

However we also see that our amplitudes, as well as our periods, respond very differently. This happens for many reasons. Ultimately it occurred because we were unable to make the proportions of our circuit elements the same as the proportions of our suspension elements like we originally hoped.

Conclusion

We have attempted to model the suspension of a 1948 Ford F-2 by using an RLC circuit. This proved to be especially difficult because the components of the suspension required us to use circuit elements that were not reasonable. As a result we had to settle for making a model that would simply give us an idea of how long the suspension would oscillate before reaching equilibrium.

The other flaw in our circuit is that it changed the resistance instead of the capacitance when the voltage across the capacitor went over a given threshold voltage. This resulted in a change of α instead of ω which didn't allow the period of the output signal to change as desired. This was done because the switch we used had a resistance associated with it that would have affected our reading on the voltage across the capacitor. With more time and resources it would be worthwhile to try to rebuild the circuit in a manner that either changed the capacitance, or if that were not possible, to rebuild it so that the inductance, in addition to the resistance, changed. This would then give us a different ω, which would cause the period to change.

It may also be better to use a more modern vehicle with a better thought-out suspension. This may provide us with values that would transfer over to circuit elements easier.

Figures and Graphs

Figure 1:

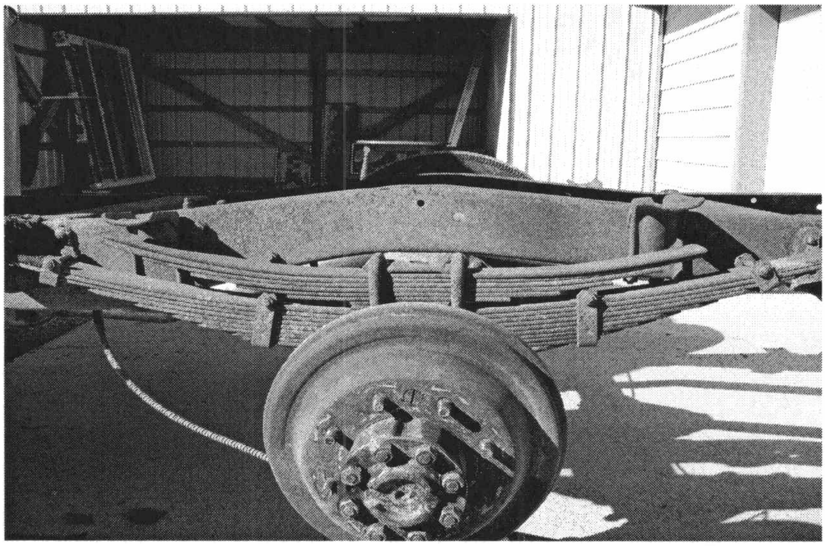


Figure 2:

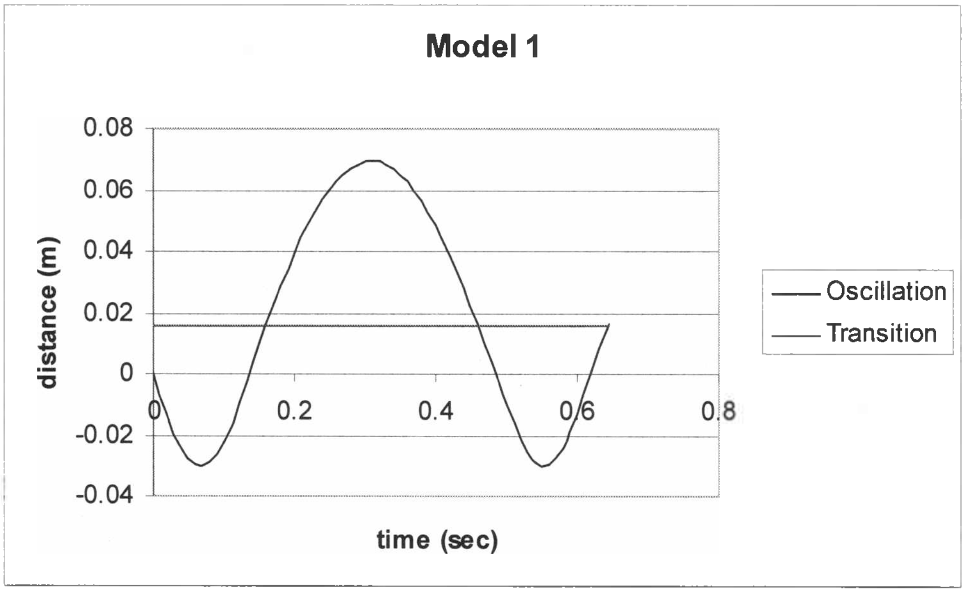


Figure 3:

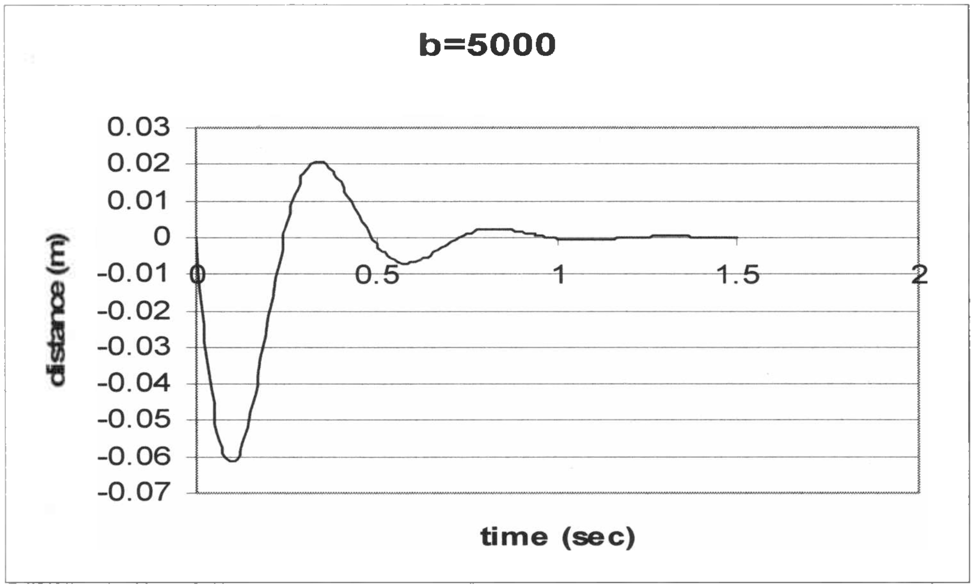


Figure 4:

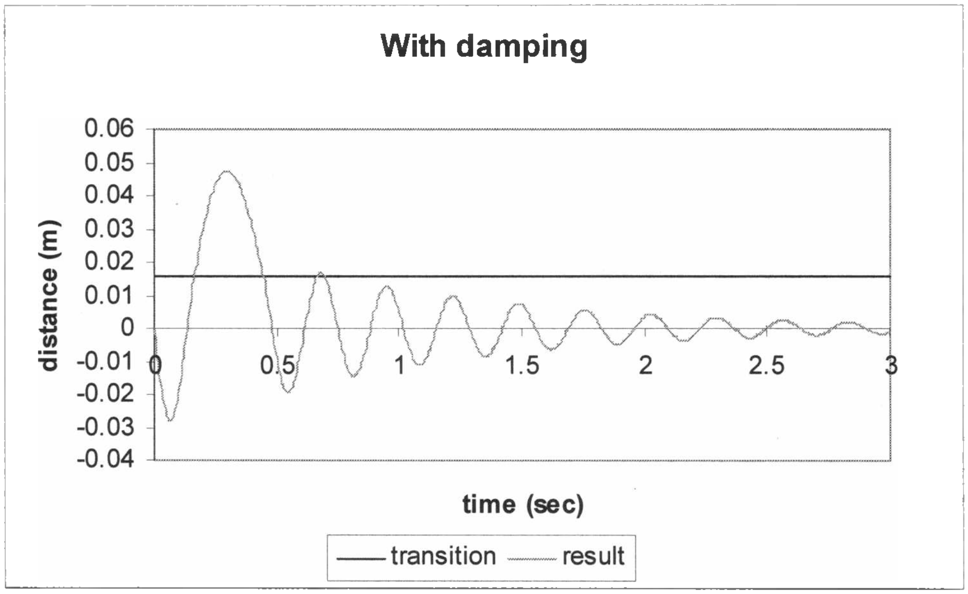


Figure 5:

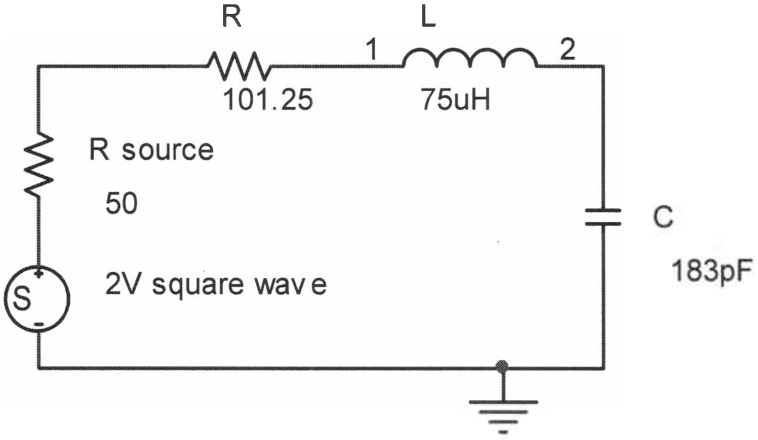


Figure 6:

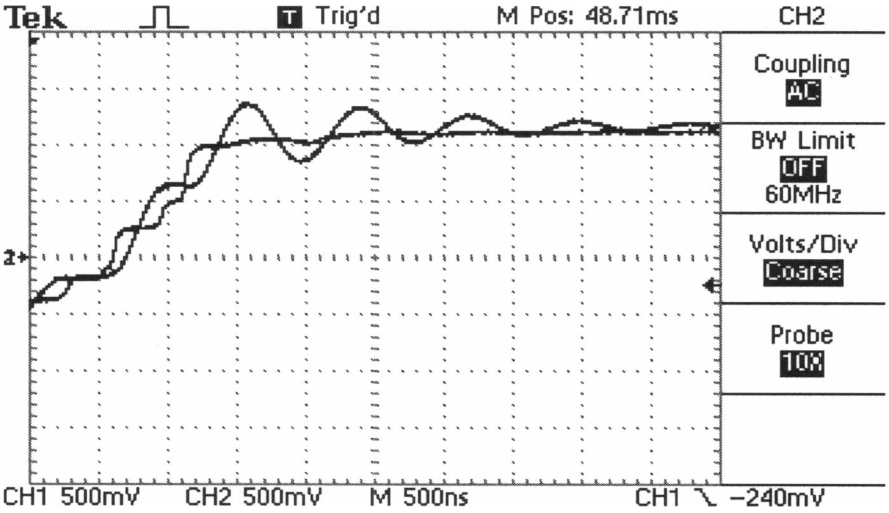


Figure 7:

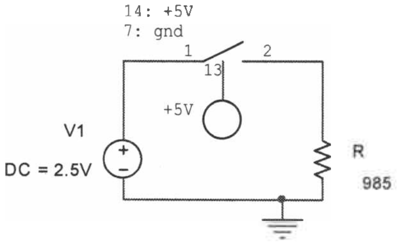


Figure 8:

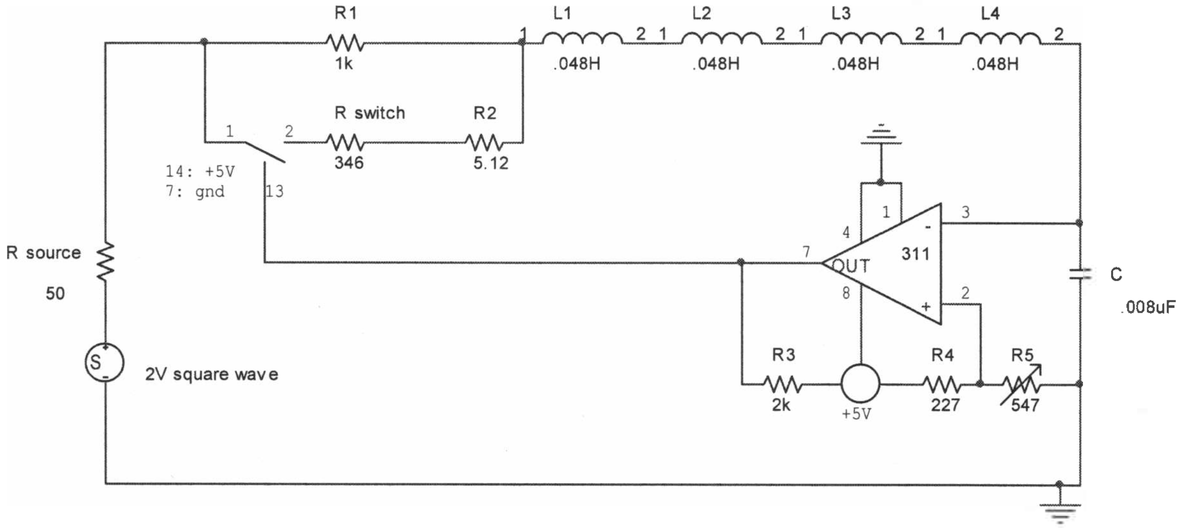
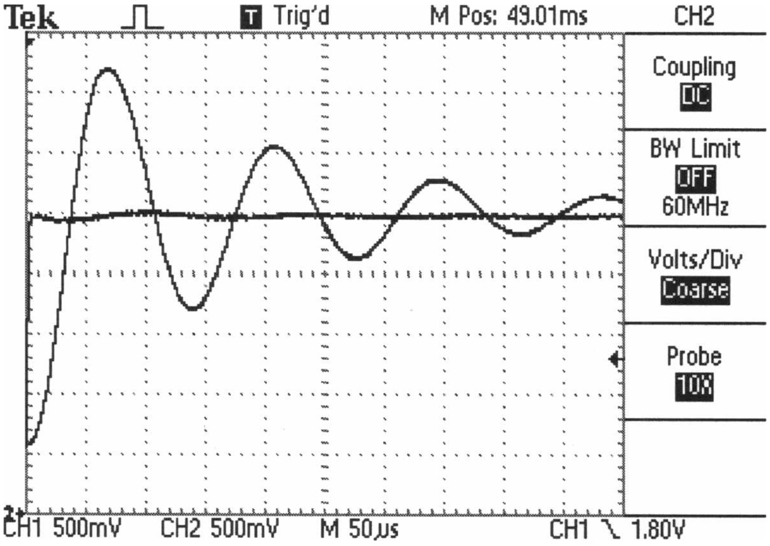


Figure 9:

A:



CH2

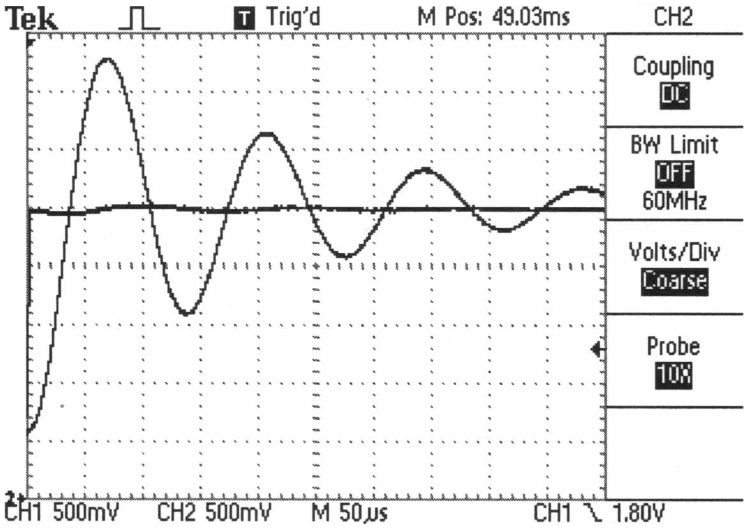
Coupling
DC

BW Limit
OFF
60MHz

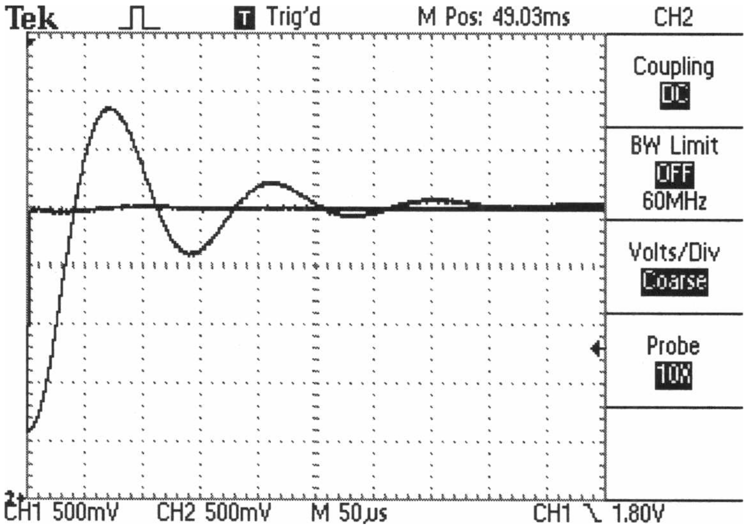
Volts/Div
Coarse

Probe
10X

B:

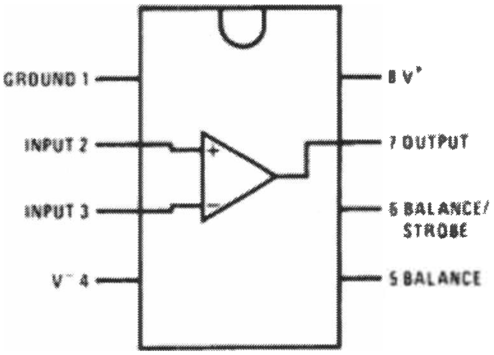

Tek
Trig'd
M Pos: 49.03ms
CH2
Coupling DC
BW Limit OFF 60MHz
Volts/Div Coarse
Probe 10X
CH1 500mV
CH2 500mV
M 50µs
CH1 ╲ 1.80V

C:

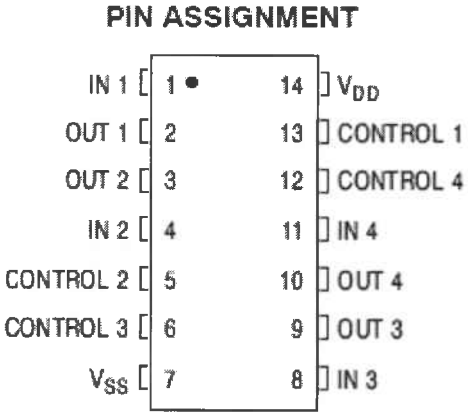

Tek
Trig'd
M Pos: 49.03ms
CH2
Coupling DC
BW Limit OFF 60MHz
Volts/Div Coarse
Probe 10X
CH1 500mV
CH2 500mV
M 50µs
CH1 ╲ 1.80V

Appendix A

Pin Assignment for LM 311



Pin Assignment for MC 14016B



Bibliography

Dixon, John. The Shock Absorber Handbook. Warrendale: Society of Automotive Engineers, Inc, 1999

Halliday, David, Robert Resnick, Jearl Walker. Fundamentals of Physics. 7th ed. Hoboken: John Wiley & Sons, 2005. 2 vols.

Irwin, David J., R. Mark Nelms. Basic Engineering Circuit Analysis. 8th ed. Hoboken: John Wiley & Sons, 2005.

Rizzoni, Giorgio. Electrical Engineering. 5th ed. New York: McGraw-Hill Companies, Inc, 2007.

Quad Analog Switch/Quad Multiplexer. OnSemiconductors. 2007. 20 Jan. 2007 <http://www.onsemi.com/pub/Collateral/MC14016B-D.PDF>.

LM 311 Voltage Comparator. National Semiconductor. 2007. 20 Jan. 2007 <http://www.national.com/pf/LM/LM311.html>

Szpilka, Anthony. “Circuit Analysis I.” Carroll College, Helena. 21 Aug. 2006.

Szpilka, Anthony. “Circuit Analysis I.” Carroll College, Helena. 11 Nov. 2006.