The Electromagnetic Wave Equations of James Clerk Maxwell

Patrick Kelly
Carroll College
The Electromagnetic Wave Equations
of James Clerk Maxwell

by
Patrick Robert Kelly

In
Partial fulfillment
of the Requirements for the
Degree of Bachelor of Science

Carroll College
Helena, Montana
April 1, 1951
INTRODUCTION

The purpose of this paper is to give an elementary presentation of Maxwell's Wave Equations. It is not the aim of the author to present any new information and the sources of this paper are as follows:

The historical data is taken almost entirely from the book "James Clerk Maxwell and Modern Physics" by Glazebrook except for a few details from "Introduction to Modern Physics" by Richtmyer and Kennard.

The field equations, the method of derivation from the differential standpoint and the discussion of the wave equations are entirely from "Principles of Electricity" by Page and Adams, except for the references to Gauss' and Stokes' Theorems.

These Theorems, together with the vector derivation are from "Introduction to Theoretical Physics" by Page.

Neither of these derivations is the same as that given by Maxwell, but are rather so arranged as to give a good introduction to the subject while involving a minimum of confusion.

To facilitate reading the paper is divided into four sections. The first deals with the life of Maxwell and the development of electrical thought prior to the time of Maxwell. The second section deals with the approach of Maxwell to the problem. The third section contains the derivation of the wave equations from the differential form of the field equations and a short discussion of the wave equations. The fourth section contains the derivation of the wave equations by vector means.
HISTORICAL BACKGROUND

The experimental and theoretical basis for the work of Maxwell was laid in the half century before his birth. In 1785 Coulomb discovered the inverse square law of attraction and repulsion and in 1800 Volta produced the first storage cell. In the first decade of the Nineteenth Century Laplace and Poisson began to apply the methods of mathematical analysis to electrostatic problems. Oersted, in 1819 discovered the magnetic field produced by an electrical current, and the next year Ampere gave a mathematical formulation for the force exerted by an electric current on a magnetic pole or on another current. Green, Gauss, Sturm and others furthered Poisson's work in electrostatics, and in 1831, the year of Maxwell's birth, Michael Faraday began his researches into electromagnetic induction.

Thus James Clerk Maxwell was born at a time when the scientific world was awakening to the challenge of electricity and magnetism. He began his formal education at the age of 10 and before he was 15 the Royal Society of Edinburgh published his first scientific paper. In 1847 he entered the University of Edinburgh and in 1850 transferred to Cambridge, where he studied until 1856. In 1855 and 1856 he published his first great electrical paper on Faraday's "Lines of Force." In the fall of 1856 he became professor of Natural Philosophy at Marischal College, Aberdeen Scotland. He moved to Kings College, London in 1860 and remained there until 1865. It was while at
Kings College that he published his greatest paper, that on "A Dynamical Theory of the Electro-Magnetic field" in 1864.

In 1856 he resigned the professorship at Kings and retired to his family home in Scotland to work on his electrical theory, which he published in "Electricity and Magnetism" in 1873. In 1871 he was appointed professor of Experimental Physics at Cambridge where he built the Cavendish Laboratory and remained as its director until his death in 1879.

The work of Maxwell spans what is perhaps the most productive period in electrical thought. Contemporary with the later works of Faraday, Gauss and Ampere, he expanded and correlated their discoveries into one inclusive theory. The expansion and experimental verification of his theory occupied physics for another 20 years after his death.
The work of Faraday is very intimately connected with that of Maxwell. Faraday produced the experimental evidence which Maxwell used in the formulation of his theory.

Up to the time of Faraday the speculation as to the cause of electrical and magnetic phenomena had centered on the Newtonian concept of "action at a distance" taking place across the space separating the bodies. In 1837 Faraday started the revolt against this idea with the publication of a paper "Electrostatic Induction" in which he showed that the force between two charged bodies depends on the medium surrounding them as well as upon their shape and position. He considered electric and magnetic induction as taking place along curved "lines of force," which he pictured as ropes of molecules starting on the conductor or magnet and acting on nearby bodies, the ropes tending to shorten and at the same time to swell laterally.

In 1855 Maxwell made this the subject of his first great electrical paper. He realized even at this early time the need for an inclusive theory of magnetism and electricity. He stated in the introduction to this paper:

"No electrical theory can now be put forth, unless it shows the connection, not only between electricity at rest and current electricity, but between the attractions and inductive effects of electricity in both states. Such a theory must accurately satisfy those laws, the mathematical form of which
is known, and must afford the means of calculating the effects in the limiting cases where the known formulae are inapplicable.*

He tried to explain the simpler phenomena of electricity and magnetism through an analogy to the flow of an incompressible liquid, the direction being due to the lines of force and the intensity due to the velocity of the liquid.

In part two of this paper which he published in February of 1856, he explained how to get a mathematical formulation of Faraday's "Electro-tonic State," (i.e.) the state into which all bodies are thrown by the presence of magnets or currents, by the definition of a "Vector Potential."

In his later papers ** Maxwell tried to devise a physical model which would enable him to explain the forces exerted on electrified bodies by means of action between the contiguous parts of the medium surrounding the bodies rather than in terms of "action at a distance."** Still basing his thoughts on "lines of force" he at one time likened the tension along the lines of force and the pressure at right angles to these lines to the contracting fibers of a muscle. To explain the pressure and tension, in which form he considered magnetic action to exist, by a physical model he needed to assume the existence of a medium with certain properties. It must be capable of exerting force on material bodies, of withstanding both tension and pressure stresses and also be capable of motion. Then all electric and magnetic

* Maxwell - "Scientific Papers" - Vol. 1, p. 155

phenomena arise from the motion and stresses of this medium.

In his model, chains of spherical cells filled with an incompressible liquid rotate about the lines of force thus being shortened in an axial direction and being expanded perpendicular to the axis of rotation by the action of centrifugal force. This would explain both the tension and the pressure stresses, and if the angular velocity of rotation is taken as a measure of the magnetic force, both the pressure and the tension can be shown to be proportional to the square of the magnetic force as experiment had shown. Since the rotation about each line of force is in the same direction the two rotating chains could not touch each other, so Maxwell postulated a continuous chain of particles which would act the part of idling wheels between the two gear trains. If the velocities of the two gear (magnetic) trains were the same the idling wheels simply rotate. On the other hand if the velocities were different the idling wheels would be forced onward in the direction of the axis with a velocity which would depend upon the difference in angular velocity of the two gear trains. Maxwell thought of these idling wheels as particles of electricity and their motion as a current. In this way he could explain the absence of induced current in a uniform magnetic field and its presence in a varying field.

To explain the difference between a dielectric and a conductor he assumed that in the conductor the particles could pass freely from molecule to molecule, while in the dielectric the medium had a certain elasticity such that the particles could be displaced only within the molecule to which they were attached.
Within such an elastic medium he noted that a wave motion could be set up which would have a velocity varying with the change in electrical properties, and he was able to show that this velocity was, within the limits of experimental error, equal to the velocity of light.

Of this mechanical model he said:

"The attempt which I then made to imagine a working model of this mechanism must be taken for no more than it really is, a demonstration that mechanism may be imagined capable of producing a connection mechanically equivalent to the actual connection of the parts of the electromagnetic field."

In "The Dynamical Theory of the Electromagnetic Field" where he abandoned all mechanical models and confined himself more to strict mathematical terms, he stated:

"All energy is the same as mechanical energy, whether it exists in the form of motion or in that of elasticity, or in any other form.

The energy in electro-magnetic phenomena is mechanical energy. The only question is, where does it reside? On the old theories it resides in the electrified bodies, conducting circuits, and magnets, in the form of an unknown quality called potential energy, or the power of producing certain effects at a distance. On our theory it resides in the electromagnetic field, in the space surrounding the electrified and magnetic bodies, as well as in those bodies themselves, and is in two different
forms, which may be described without hypothesis as magnetic polarization and electric polarization or, according to a very probably hypothesis, as the motion and the strain of one and the same medium.*

Maxwell conceived the field as a complicated mechanism which was capable of a great variety of motion but one in which the motion of one part was dependent upon the motion of the other parts, and one which should be subject to the general laws of dynamics for any connected system.

From a study of condensors and their effect on the dielectric between their plates he reached the conclusion that the variation of the electric displacement is equivalent in all respects to a current, and that the current at any point in a dielectric is measured by the rate of change of displacement at that point. He then had the four relations

(1) Between electric force and electric current in a conductor.

(2) Between electric force and electric displacement in a dielectric.

(3) Between electric force and the changes of magnetic induction which give rise to it.

(4) Between magnetic force and the current which gives rise to it.

from which he obtained his field equations.

* Philosophical Transactions - 1864
The four equations with which Maxwell had to work described the electromagnetic field produced by charges at rest or in motion. These equations, given by earlier experimenters, were

(1) Gauss' Law specifying the flux of displacement through a closed surface surrounding a quantity of free charge of density $s$.

\[ \oint_S D \cos \theta \, ds = 4\pi \int_V s \, dv \]  

(1)

Or in vector form

\[ \oint_S D \cdot ds = 4\pi \int_V s \, dv \]  

(1a)

Or by Gauss' Theorem since

\[ \oint_S D \cdot ds = \int_V \nabla \cdot D \, dv \]  

\[ \nabla \cdot D = 4\pi s \]  

(1aa)

(2) Gauss' Law specifying the flux of induction through a closed surface surrounding a magnetic medium.

\[ \oint_S B \cos \theta \, ds = 0 \]  

(2)

Or in vector form

\[ \oint_S B \cdot ds = 0 \]  

(2a)
Or by Gauss' Theorem

\[ \nabla \cdot B = 0 \]  \hspace{1cm} (2aa)

(3) Ampere's Law specifying the magnetic field due to a current

\[ \oint H \cos \theta \, dl = \frac{2\pi j}{c} \]  \hspace{1cm} (3)

Or in vector form

\[ \oint H \cdot dl = \frac{2\pi j}{c} \int j \cdot ds \]  \hspace{1cm} (3a)

Where \( j \) is the current density over the surface

or by Stokes' Theorem since

\[ \oint H \cdot dl = \int (\nabla \times H) \cdot ds \]

\[ \nabla \times H = \frac{2\pi j}{c} \]  \hspace{1cm} (3aa)

(h) Faraday's Law specifying the electric field produced by a changing magnetic flux

\[ \oint E \cos \theta \, dl = -\frac{1}{c} \int \frac{\partial}{\partial t} (B \cos \theta) \, ds \]  \hspace{1cm} (h)

Or in vector form

\[ \oint E \cdot dl = -\frac{1}{c} \int \frac{\partial B}{\partial t} \cdot ds \]  \hspace{1cm} (ha)
Or by Stokes' Theorem

\[ \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} \]

Two other relations are needed

\[ D = E + \varepsilon_0 \rho \]  \hspace{1cm} (5)

Or for isotropic media

\[ D = \kappa E \]  \hspace{1cm} (5a)

and

\[ B = H + \mu_0 I \]  \hspace{1cm} (6)

Or for isotropic paramagnetic media

\[ B = \mu_0 H \]  \hspace{1cm} (6a)

The key contribution by Maxwell to these equations was

his realization that they would not satisfy the equation of

continuity

\[ \frac{\partial E}{\partial t} + \frac{\partial H}{\partial x} + \frac{\partial \varepsilon_0}{\partial y} + \frac{\partial \mu_0}{\partial z} = 0 \]

Where \( V \) is the amount of charge flowing into a certain volume and \( \frac{\partial \varepsilon_0}{\partial t} \) is the time rate of increase of charge inside the volume. By correcting Ampere's law he arrived at a system of equations which would give the same results as the uncorrected equations for static fields and would at the same time satisfy the equation of continuity, Maxwell called this added factor a "displacement current." When the correction was made Ampere's law took the form
From this point the derivation of the wave equations will follow two lines of thought. First, the equations of a wave motion will be derived from the differential form of the field equations and these wave equations will be briefly discussed. Secondly, the equations of the same wave motion will be derived from the vector form of the field equations and the results will be interpreted in terms of vectors.
DERIVATION OF THE WAVE EQUATIONS FROM THE
DIFFERENTIAL FORM OF THE FIELD EQUATIONS.

D, E, B and H are functions of \((x, y, z, t)\), continuous with
their first partials in the region under consideration.
Therefore

\[
\begin{align*}
D_{ij} &= D_{ji} \quad E_{ij} = E_{ji} \quad B_{ij} = B_{ji} \quad H_{ij} = H_{ji}
\end{align*}
\]

The Field Equations are easily given in Differential Form.
Considering a cube of dimensions \(\Delta x, \Delta y, \Delta z\) as the unit of
volume Gauss' law (1) for electric flux through a closed surface gives

\[
\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 4\pi
\]

(1')

Over the same volume Gauss' law (2) for flux of induction through
a closed surface gives

\[
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0
\]

(2')

Considering the edges of a surface of this cube as a circuit
Ampere's law specifying the magnetic field due to a current gives,
for the three surfaces passing through the origin, the three
scalar equations

\[
\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} = \mu \left[ \frac{\partial D_x}{\partial t} + \nabla \cdot \mathbf{J} \right] \quad (3'-a)
\]

\[
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \mu \left[ \frac{\partial D_y}{\partial t} + \nabla \cdot \mathbf{J} \right] \quad (3'-b)
\]

\[
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \mu \left[ \frac{\partial D_z}{\partial t} + \nabla \cdot \mathbf{J} \right] \quad (3'-c)
\]
over the same three circuits Faraday's law \((h)\) for the electric field produced by a changing magnetic flux gives the three scalar equations

\[
\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c} \frac{\partial B_z}{\partial t} \tag{h'-a}
\]

\[
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{1}{c} \frac{\partial B_y}{\partial t} \tag{h'-b}
\]

\[
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{1}{c} \frac{\partial B_z}{\partial t} \tag{h'-c}
\]

If the medium under consideration is homogeneous and isotropic, and contains neither current nor free charge

\[D = \kappa E \tag{7}\]

and

\[B = \mu H \tag{8}\]

and

\[j = 0 \quad \rho = 0 \]

the Differential Form of the Field Equations becomes

\[
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{1''}
\]

\[
\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \tag{2''}
\]
from these equations a wave equation in the components of $\mathbf{E}$ can be obtained by eliminating the $\mathbf{H}$ and an exactly similar equation in the components of $\mathbf{H}$ can be obtained by eliminating the $\mathbf{E}$ components.
A) Differentiating (3"a) with respect to the time

\[ \frac{\partial^2 P}{\partial t \partial x} - \frac{\partial^2 P}{\partial t \partial y} = \frac{\partial}{\partial t} \frac{\partial P}{\partial t} \]  

\[ (3''at) \]

Differentiating (4"b) with respect to z

\[ \frac{\partial^2 E}{\partial z \partial x} - \frac{\partial^2 E}{\partial z \partial y} = -\frac{\partial}{\partial z} \frac{\partial E}{\partial z} \]  

\[ (4''bz) \]

Differentiating (4"c) with respect to y

\[ \frac{\partial^2 H}{\partial y \partial x} - \frac{\partial^2 H}{\partial y \partial z} = -\frac{\partial}{\partial y} \frac{\partial H}{\partial y} \]  

\[ (4''cy) \]

Substituting \[ \ldots \] since the first partials of H and E are continuous functions of (x, y, z, t)

\[ H_{ij} = H_{ji} \quad E_{ij} = E_{ji} \]

\[ \frac{\partial^2 E}{\partial y \partial x} + \frac{\partial^2 E}{\partial z \partial x} - \frac{\partial}{\partial x} \left[ \frac{\partial^2 E}{\partial y \partial x} + \frac{\partial^2 E}{\partial z \partial x} \right] = \frac{\partial}{\partial t} \frac{\partial^2 E}{\partial t \partial t} \]

or

\[ \frac{\partial^2 E}{\partial y \partial x} + \frac{\partial^2 E}{\partial z \partial x} - \frac{\partial}{\partial x} \left[ \frac{\partial^2 E}{\partial y \partial x} + \frac{\partial^2 E}{\partial z \partial x} \right] = \frac{\partial}{\partial t} \frac{\partial^2 E}{\partial t \partial t} \]

but from (1")

\[ \frac{\partial E}{\partial y} + \frac{\partial E}{\partial z} = -\frac{\partial E}{\partial x} \]

\[ (1'') \]

Therefore

\[ \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \frac{\partial}{\partial t} \frac{\partial^2 E}{\partial t \partial t} \]

\[ (A) \]
B) Differentiating (3"b) with respect to the time

\[ \frac{\partial^2 H}{\partial t^2} - \frac{\partial^2 H}{\partial t \partial x} = \frac{\mu}{\alpha} \frac{\partial^2 E_y}{\partial x^2} \quad (3"bt) \]

Differentiating (4"a) with respect to z

\[ \frac{\partial^2 E_z}{\partial z \partial t} - \frac{\partial^2 E_z}{\partial t^2} = -\mu \frac{\partial^2 H}{\partial x \partial t} \quad (4"az) \]

Differentiating (4"c) with respect to x

\[ \frac{\partial^2 E_x}{\partial x^2} - \frac{\partial^2 E_x}{\partial x \partial t} = -\mu \frac{\partial^2 H}{\partial x \partial t} \quad (4"cx) \]

Substituting since the first partials of H and E are continuous functions of (x, y, z, t)

\[ H_{ij} = H_{ji} \quad E_{ij} = E_{ji} \]

\[ \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} - \left[ \frac{\partial^2 E_x}{\partial x \partial y} + \frac{\partial^2 E_z}{\partial x \partial z} \right] = \frac{\mu \chi}{\alpha^2} \frac{\partial^2 E_y}{\partial t^2} \]

or

\[ \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} - \frac{\partial}{\partial y} \left[ \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} \right] = \frac{\mu \chi}{\alpha^2} \frac{\partial^2 E_y}{\partial t^2} \]

but from (1"n)

\[ \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = -\frac{\partial E_y}{\partial y} \quad (1"n) \]

Therefore

\[ \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \frac{\mu \chi}{\alpha^2} \frac{\partial^2 E_y}{\partial t^2} \quad (B) \]
C) Differentiating \( (3^{\text{nd}}) \) with respect to the time

\[
\frac{\partial^2 H}{\partial t^2} \frac{\partial x}{\partial t} = \frac{\partial^2 E_x}{\partial t^2} \quad (3^{\text{nd}}t)
\]

Differentiating \( (4^{\text{th}}a) \) with respect to \( y \)

\[
\frac{\partial^2 E_x}{\partial y^2} - \frac{\partial^2 E_y}{\partial y^2} = -\mu \frac{\partial^2 H}{\partial y \partial t} \quad (4^{\text{th}}a\gamma)
\]

Differentiating \( (4^{\text{th}}b) \) with respect to \( x \)

\[
\frac{\partial^2 E_x}{\partial x \partial z} - \frac{\partial^2 E_z}{\partial x \partial z} = -\mu \frac{\partial^2 H}{\partial x \partial t} \quad (4^{\text{th}}b\delta)
\]

Substituting \( \ldots \) since the first partials of \( H \) and \( E \) are continuous functions of \( (x, y, z, t) \)

\[
H_{ij} = H_{ji} \quad E_{ij} = E_{ji}
\]

or

\[
\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - \frac{\partial^2 E_y}{\partial y \partial t} = \frac{k \mu}{c^2} \frac{\partial^2 E_x}{\partial t^2}
\]

but from \( (1^{\text{st}}) \)

\[
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = -\frac{\partial E_z}{\partial z} \quad (1^{\text{st}})
\]

Therefore

\[
\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = \frac{\mu k \omega^2}{c^2} \frac{\partial^2 E_x}{\partial t^2} \quad (0)
\]
D) Differentiating \( \frac{\partial^2 E_t}{\partial t \partial y} \) with respect to the time

\[
\frac{\partial^2 E_t}{\partial t \partial y} - \frac{\partial^2 E_y}{\partial y \partial z} = -\mu \frac{\partial^2 H_x}{\partial t \partial \zeta^2}
\]  

(4ata)

Differentiating (3nb) with respect to \( z \)

\[
\frac{\partial^2 H_x}{\partial z \partial x} - \frac{\partial^2 H_z}{\partial z \partial \zeta} = \kappa \frac{\partial^2 E_y}{\partial z \partial \zeta}
\]  

(3nbz)

Differentiating (3nc) with respect to \( y \)

\[
\frac{\partial^2 H_y}{\partial y \partial x} - \frac{\partial^2 H_y}{\partial y \partial z} = \kappa \frac{\partial^2 E_z}{\partial y \partial \zeta}
\]  

(3ncy)

Substituting —— since the first partials of \( H \) and \( E \) are continuous functions of \( (x, y, z, t) \)

\[
\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} - \left[ \frac{\partial^2 H_y}{\partial y \partial x} + \frac{\partial^2 H_z}{\partial z \partial x} \right] = \frac{\mu \kappa}{\sigma^2} \frac{\partial^2 H_x}{\partial t \partial \zeta^2}
\]

since

\( E_{ij} = E_{ji} \)

\( H_{ij} = H_{ji} \)

or

\[
\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} - \left[ \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} \right] = \frac{\mu \kappa}{\sigma^2} \frac{\partial^2 H_x}{\partial t \partial \zeta^2}
\]

but from (2n)

\[
\frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = -\frac{\partial H_x}{\partial x}
\]  

(2n)

Therefore

\[
\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} = \frac{\mu \kappa}{\sigma^2} \frac{\partial^2 H_x}{\partial t \partial \zeta^2}
\]  

(D)
E) Differentiating \((h_1b)\) with respect to the time

\[
\frac{\partial^2 E_x}{\partial t^2} + \frac{\partial^2 E_x}{\partial x^2} = -\mu_x \frac{\partial^2 H_y}{\partial t^2} \tag{1^\text{st}t}
\]

Differentiating \((3^\text{a})\) with respect to \(z\)

\[
\frac{\partial^2 H_x}{\partial z^2} = \varepsilon \frac{\partial^2 E_x}{\partial t^2} \tag{3^\text{a}z}
\]

Differentiating \((3^\text{c})\) with respect to \(x\)

\[
\frac{\partial^2 H_y}{\partial x^2} = \varepsilon \frac{\partial^2 E_x}{\partial t^2} \tag{3^\text{c}x}
\]

Substituting \(-\varepsilon\) since the first partials of \(H\) and \(E\) are continuous functions of \((x, y, z, t)\)

\[
E_{ij} = E_{ji} \quad H_{ij} = H_{ji}
\]

\[
\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} - \left[\frac{\partial^2 H_x}{\partial z^2} + \frac{\partial^2 H_x}{\partial x^2}\right] = \frac{\mu_x}{\varepsilon^2} \frac{\partial^2 H_y}{\partial t^2}
\]

or

\[
\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} - \frac{\partial}{\partial y} \left[\frac{\partial H_x}{\partial x} + \frac{\partial H_x}{\partial z}\right] = \frac{\mu_x}{\varepsilon^2} \frac{\partial^2 H_y}{\partial t^2}
\]

but from \((2^\text{n})\)

\[
\frac{\partial H_x}{\partial x} + \frac{\partial H_x}{\partial z} = -\frac{\partial H_y}{\partial y} \tag{2^\text{n}}
\]

Therefore

\[
\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} = \frac{\mu_x}{\varepsilon^2} \frac{\partial^2 H_y}{\partial t^2} \tag{E}
\]
F) Differentiating \( \frac{\partial^2 E_y}{\partial t^2} - \frac{\partial^2 E_y}{\partial t \partial y} = -\alpha^2 \frac{\partial^2 H_y}{\partial x^2} \) (4"ct)

Differentiating \( \frac{\partial^2 H_z}{\partial y^2} - \frac{\partial^2 H_z}{\partial y \partial z} = \frac{\mu}{c} \frac{\partial^2 E_y}{\partial y \partial t} \) (3"ay)

Differentiating \( \frac{\partial^2 H_x}{\partial x^2} - \frac{\partial^2 H_x}{\partial x \partial z} = \frac{\mu}{c} \frac{\partial^2 E_y}{\partial x \partial t} \) (3"bx)

Substituting since the first partials of \( H \) and \( E \) are continuous functions of \( (x, y, z, t) \)

\[ E_{ij} = E_{ji} \quad H_{ij} = H_{ji} \]

\[ \frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} = \left[ \frac{\partial^2 H_x}{\partial x \partial z} + \frac{\partial^2 H_x}{\partial y \partial z} \right] = \frac{\mu}{c^2} \frac{\partial^2 H_x}{\partial z^2} \]

or

\[ \frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} = \left[ \frac{\partial^2 H_x}{\partial x \partial z} + \frac{\partial^2 H_x}{\partial y \partial z} \right] = \frac{\mu}{c^2} \frac{\partial^2 H_x}{\partial z^2} \]

but from (2"

\[ \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = -\frac{\partial H_z}{\partial z} \] (2"

Therefore

\[ \frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} = \frac{\mu}{c^2} \frac{\partial^2 H_x}{\partial z^2} \]
To discuss the characteristics of these equations we will first consider the case where $E$ and $H$ are functions of $(x, t)$ only. In this case the wave equations reduce to

\[ \frac{\partial^2 E_x}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 E_x}{\partial t^2} \]  
(A-a)

\[ \frac{\partial^2 E_y}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 E_y}{\partial t^2} \]  
(B-a)

\[ \frac{\partial^2 E_z}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 E_z}{\partial t^2} \]  
(C-a)

and

\[ \frac{\partial^2 H_x}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 H_x}{\partial t^2} \]  
(D-a)

\[ \frac{\partial^2 H_y}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 H_y}{\partial t^2} \]  
(E-a)

\[ \frac{\partial^2 H_z}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 H_z}{\partial t^2} \]  
(F-a)

where

\[ \nu^2 = \frac{c^2}{\mu \kappa} \]  
(9)

the general solution of each of these equations is of the type

\[ E_x = \phi_1(x-\nu t) + \phi_2(x-\nu t) \]
where $\phi_1$ is any differentiable function.

The first term represents a plane wave advancing along the positive x axis with a velocity $v$. It is a plane because, not being a function of $y$ or $z$, it has the same value at a given instant at all points with the same x coordinate. Since

$$\phi_1 \left[ (x+vt) - v(t+\tau) \right] = \phi_1 (x-vt)$$

the function has the same value at the point $(x+vt)$ and the time $(t+\tau)$ as it has at $(x,t)$ it represents a wave with phase velocity $v$.

The second term represents a wave traveling along the negative x axis with the same velocity.

We will consider only the waves moving in the positive x direction. Then

$$E_x = 0 (x-vt) \quad H_x = f(x-vt)$$

$$E_y = 0 (x-vt) \quad H_y = g(x-vt)$$

$$E_z = 0 (x-vt) \quad H_z = h(x-vt)$$

These equations must satisfy equations (1**, 2**, 3**, 4**) and this fact gives us an easy way to discover more about them. From (1**)

$$\frac{\partial E_x}{\partial x} = 0$$

Since $E$ is not a function of $y$ or $z$. Therefore $E$ is a constant in $x$, and we can assume this constant to be 0 since the static electric field is of no interest. Since

$$\frac{\partial E_x}{\partial t} = -v \frac{\partial E_x}{\partial x}$$
E is also constant in t. From (2n)

$$\frac{\partial H}{\partial x} = 0$$

and H is therefore constant in x and t, since

$$\frac{\partial H}{\partial t} = - \nu \frac{\partial H}{\partial x}$$

Since E and H are constant with respect to x they are both perpendicular to the direction of propagation and therefore our waves are transverse.

From (3n)b)

$$\frac{dH_y}{d(x-ut)} = \nu k \frac{dE_z}{d(x-ut)}$$

or on integrating

$$H_y = \sqrt{\frac{k}{\mu}} E_z + C$$

but as our interest is in the wave alone we can set C equal to 0

From (3n)c)

$$\frac{dH_y}{d(x-ut)} = - \nu k \frac{dE_z}{d(x-ut)}$$

or on integrating and setting the constant equal to 0

$$H_y = - \sqrt{\frac{k}{\mu}} E_z$$

the equations of two plane, transverse waves advancing along the positive x axis with a phase velocity v take the form

$$E_x = 0 \quad H_x = 0$$

$$E_y = \Theta(x-ut) \quad H_y = - \sqrt{\frac{k}{\mu}} E_z$$

$$E_z = \Phi(x-ut) \quad H_z = \sqrt{\frac{k}{\mu}} E_y$$
the dot product of two vectors is defined as

\[ E \cdot H = EH \cos \theta = E_x H_x + E_y H_y + E_z H_z \]

where \( \theta \) is the angle between the vectors but

\[ E \cdot H = \sqrt{E^2} \sqrt{H^2} \cos \theta = 0 \]

therefore \( E \) and \( H \) lie in the wave front at right angles to each other.

In the more general case where \( E \) and \( H \) are functions of \((x,y,z,t)\) the solutions are of the type

\[ E_x = \phi (\alpha x + \beta y + \gamma z - \nu t) \]

or the equation of a plane wave advancing with phase velocity \( \nu \) along the line

\[ \frac{x-x_0}{\alpha} = \frac{y-y_0}{\beta} = \frac{z-z_0}{\gamma} \]

In a vacuum \( \mu, \varepsilon \) are substantially equal to 1. Therefore

\[ \nu^2 = c^2 \quad (9a) \]

From the last conclusion and the fact that electromagnetic waves are plane and transverse, the inference that light is an electromagnetic phenomenon is practically inescapable.
DERIVATION OF THE WAVE EQUATIONS FROM THE VECTOR FORM OF THE FIELD EQUATIONS

The derivation by vector means is relatively easy. For a homogeneous isotropic medium with no current or free charge since

\[ \Phi = 0 \quad J = 0 \quad B = \mu H \quad D = \varepsilon E \]

The vector field equations become

\[ \nabla \cdot E = 0 \quad (1^\prime \text{aa}) \]
\[ \nabla \cdot H = 0 \quad (2^\prime \text{aa}) \]
\[ \nabla \times H = \frac{\partial E}{\partial t} \quad (3^\prime \text{cc}) \]
\[ \nabla \times E = -\varepsilon \frac{\partial H}{\partial t} \quad (4^\prime \text{aa}) \]

Where \( \dot{A} \) is the partial of \( A \) with respect to the time.

Taking the curl of \( (3^\prime \text{cc}) \)

\[ \nabla \times (\nabla \times H) = \mu \nabla \times \dot{E} \]

Taking the partial of \( (4^\prime \text{aa}) \) with respect to the time

\[ \frac{\partial}{\partial t} (\nabla \times E) = \nabla \times \dot{E} = -\varepsilon \frac{\partial H}{\partial t} \]

The triple vector product is expanded by

\[ \nabla \times (\nabla \times H) = \nabla \cdot H \nabla - \nabla \cdot \nabla H \]

but

\[ \nabla \cdot H = 0 \quad (2^\prime \text{aa}) \]

Therefore

\[ \nabla \cdot \nabla H = \frac{\mu}{\varepsilon} \dot{J} \quad (\text{AA}) \]
Taking the curl of \((\mathbf{h}_1 - \mathbf{a})\)

\[ \nabla \times (\nabla \times \mathbf{E}) = -\frac{\mu}{c^2} \nabla \times \mathbf{H} \]

Taking the partial of \((\mathbf{3} - \mathbf{c})\) with respect to the time

\[ \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) = \nabla \times \mathbf{H} = \frac{\kappa}{c} \mathbf{E} \]

but the triple vector product

\[ \nabla \times (\nabla \times \mathbf{E}) = \nabla \cdot \mathbf{E} \mathbf{V} - \nabla \cdot \nabla \mathbf{E} \]

but

\[ \nabla \cdot \mathbf{E} = 0 \]

therefore

\[ \nabla \cdot \nabla \mathbf{E} = \frac{\mu \kappa}{c^2} \mathbf{E} \]

\(^{(BB)}\)

\((AA)\) and \((BB)\) are the equations of the same wave as is given by the six scalar equations \((A, B, C, D, E, F)\).
<table>
<thead>
<tr>
<th>Author</th>
<th>Title</th>
<th>Publisher</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glazebrook</td>
<td>James Clerk Maxwell and Modern Physics</td>
<td>MacMillan</td>
<td>1896</td>
</tr>
<tr>
<td>Osgood</td>
<td>Advanced Calculus</td>
<td>MacMillian</td>
<td>1925</td>
</tr>
<tr>
<td>Page</td>
<td>Introduction to Theoretical Physics</td>
<td>Van Nostrand</td>
<td>1935</td>
</tr>
<tr>
<td>Page-Adams</td>
<td>Principles of Electricity</td>
<td>Van Nostrand</td>
<td>1949</td>
</tr>
<tr>
<td>Richtmyer and Kennard</td>
<td>Introduction to Modern Physics</td>
<td>McCraw-Hill</td>
<td>1947</td>
</tr>
</tbody>
</table>