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The Separation of a Two-Nanosatellite System via Differential Drag

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by Ian Lyon

This thesis for honors recognition has been approved for the Department of Mathematics.

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The Separation of a Two-Nanosatellite System via Differential Drag

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A thesis submitted for the degree of
Bachelor of Fine Arts

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ABSTRACT

Small nanosatellites (1 to 10 kg), with their low production costs and unique mission strategy possibilities, can be inexpensively deployed in groups to maximize scientific returns. The Space Science and Engineering Laboratory (Montana State University) has demonstrated this with their FIREBIRD satellites, a two-satellite system set to observe relativistic electron bursts in the Earth’s magnetosphere.

FIREBIRD separation strategy scenarios that use differential drag to influence relative satellite velocities are studied here. Relevant orbital mechanics concepts are discussed, and equations of motion are derived and incorporated into an orbital simulation model written in MATLAB.

It is found that a springless separation strategy utilizing differential drag induced by differing satellite masses creates a near-constant acceleration throughout the FIREBIRD mission duration. This method is found to be a feasible alternative to the traditional springed-separation strategy. At an altitude of 700 km, a mass difference of five grams will separate the FIREBIRD satellites to a maximum allowed relative distance of 100 km in roughly a year, plus or minus eight months.

Additional physics concepts which could improve the accuracy of the model are also discussed.
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CHAPTER 1

SMALL SATELLITES

Since the late 1950’s, the world has placed satellites into orbit for applications in research, communications, navigation, and the military. When many of us think of a satellite, we may imagine a large, expensive, and incredibly complex piece of technology. Indeed, it may take many years of research and development to prepare a satellite for its flight high above our planet. Once in orbit, it will begin its carefully planned long-term mission, perhaps performing several functions at once. Some of these satellites are integral to the well-being of humanity; for example, our ability to send and receive almost all information from around the world depends on our numerous communication satellites.[1]

Many important satellite projects are best implemented by the top scientists and engineers with the most advanced technology (and the greatest funding) at their fingertips. However, satellites and their missions are quickly evolving. In this modern age of innovation and advancement, our society has overseen a profound shift towards the miniaturization of technology. The world of satellites is no stranger to these changes. As our satellites get smaller, they become more accessible to an increasingly larger pool of individuals willing to reap the benefits. From the driven amateur to seasoned professional, from the researcher in a small engineering firm to the professor in prestigious academia, each can become an author in a small satellite’s success story.

Even those at the top of the “large satellite” game can see the advantages in going smaller. The National Aeronautics and Space Administration (NASA) has been quickly developing its first demonstration small satellite, called the Fast, Affordable, Science and Technology Satellite (FASTSAT). The rationale behind this is simple: a smaller satellite will require considerably less time and funding to complete, versus a standard full-scale satellite.

[1]Communications satellites often have a “geosynchronous orbit.” They fly at an altitude of almost 36,000 kilometers, or three Earth diameters.[14].
The mission of FASTSAT is to provide a quick, low-cost opportunity for researchers to perform experiments in orbit.

1.1 Why Go Smaller?

The benefits of going small do not cease with the reduced costs. NASA also recognizes the small satellite’s ability to safely test new technologies and hardware in the field. Newer aeronautics technologies are constantly introduced, and many of these may be implemented in future large satellite missions. The opportunity to test these technologies in the field by flying them on an inexpensive small satellite is truly tempting, since this creates a low-risk testing environment where more expensive satellites are not in danger[8].

Sometimes small satellites are the best choice when a mission’s success requires using something “small.” For instance, a certain satellite could use a particularly minuscule sensor to gather data for a mission. It may be advantageous for a group of two or more of these satellites to orbit in a formation, since this will allow them to gather data from multiple places at the same time. Using such a method with traditional larger satellites would create a prohibitively expensive mission, as the cost of any individual large satellite would easily exceed that of any small ones[18]. Therefore, the mission would be better suited for the use of a small satellite formation.

NASA is not alone in their reasoning behind the creation of smaller, versatile, and affordable satellites. They aren’t even the first to create them. More than four times as many satellites smaller than 5 kilograms were launched in the past ten years than in the decade before[18]. There are also several devoted organizations charged with spreading the use of small satellites for applicable missions. For 23 years, the American Institute of Aeronautics and Astronautics (AIAA), in conjunction with Utah State University (USU), has held the AIAA/USU Conference on Small Satellites. The conference is devoted to the exchange of ideas, concepts, and technologies directly related to the use of small satellites in current missions[3]. The field is expanding, and the need for efficient and inexpensive satellite projects is rising quickly.
1.1 Why Go Smaller?

Out of all the projects devoted to the introduction of “small” to the satellite world, one of the most notable is the innovative CubeSat Project. In 1999, California Polytechnic State University (Cal Poly) and Stanford University’s Space Systems Development Laboratory (SSDL) created standardized specifications for a very small satellite that could be produced by universities and corporations across the nation. The satellite type was dubbed the “CubeSat” (see Figure 1.1 on the opposite page and Figure 1.2 above). The popularity of the CubeSat increased with recent fervor over the past ten years. By 2006, there were over 30 CubeSats in orbit. More than a hundred new Cubesats will be launched within the next few years, and many of these satellites will be designed and constructed by students and professors from universities across the nation.

Each one of these CubeSats carries out a real mission, and aims to provide large contributions to science. It is important to note that these individual missions could not be implemented easily if a larger satellite were used, especially not at the university level; the required funding is just nonexistent. The fact that an entire satellite can be envisioned, designed, and built by graduate and undergraduate students (and provide useful science) gives special credence to the success of the CubeSat project.

As a student, my intrigue with the capabilities of the CubeSat is the primary driving force which guides the topics throughout this thesis. Having personally worked with...
1. SMALL SATELLITES

CubeSats\[1 I have a vested interest in explaining how crucial these satellites are to an engineering student’s education. In my experience, working with CubeSats provides unparalleled learning opportunities and opens many avenues for new research. Therefore, for the remainder of this paper, I will discuss the many facets of CubeSats, the physics of satellites (orbital mechanics), and the physics of a particular CubeSat mission. This will then lead us to our exploration of an interesting orbital mechanics problem.

1.2 CUBE SATS AND THEIR SYSTEM

1.2.1 STRUCTURE

Generally, the classification of small satellites is broken up into a few categories. Usually, a small satellite that weighs between 100 and 500 kg would be called a minisatellite. Satellites that weigh 10 to 100 kg are called microsatellites. Smaller still are nanosatellites, which usually weigh between 1 and 10 kg. Among the very smallest type of small satellite is the picosatellite, which is between about 100 g and 1 kg\[9].

When Cal Poly and SSDL originally created the specifications for a CubeSat, they had simplicity in mind. A normal CubeSat is a 10 cm cube, weighing around 1.3 kg at the most, but generally a little less than 1 kg. This tentatively places a CubeSat into the picosatellite

---

1 I have worked with the Space Science and Engineering Laboratory (SSEL) in Bozeman, Montana, during the summers of 2009 and 2010. My work with SSEL was affiliated with Montana State University and the Montana Space Grant Consortium.
1.2 CubeSats and their System

Figure 1.4: Demonstrating 1U, 2U, and 3U CubeSat dimensional units[18].

category. Their compact and simple shape allows them to easily fit into an innovative deployment mechanism called the P-POD (Poly-Picosatellite Orbital Deployer), as shown in Figure 1.3 on the opposite page.

In a P-POD, three normal CubeSats are stacked against one another within a compartment. They stay in this compartment until they are ready to be deployed into space. When this happens, a large spring mechanism pushes them all out into the desired orbital trajectory[21]. More springs placed between the individual CubeSats will then separate the satellites from one another, so they don’t bump into each other or exhibit other unintended behaviors[20]. Figure 1.5 on page 6 illustrates the CubeSats’ deployment process.

While the 10×10×10 cm CubeSat is the standard size, there are longer variations that can still fit inside the P-POD. One type is twice as long as a normal CubeSat, so that its dimensions are 20×10×10 cm. This is called a 2U CubeSat, since it contains two units (U) of 10 cm along its length. This would make the standard CubeSat with a length of 10 cm a 1U. Another type, longer still, is the 3U CubeSat; this one has a length of 30 cm. It is true that the lengths of the 2U and 3U CubeSats are very different than that of the normal 1U satellite. However, changing only the length (not the width or height) allows one to place many possible combinations of 1U, 2U, and 3U CubeSats in a P-POD. For instance, a P-POD could either hold three 1U CubeSats, one 3U CubeSat, or one 2U and a 1U. Other variations involving nonstandard sizes, such as two 1.5U CubeSats, are also entirely possible[21]. The general “U” categories are shown in Figure 1.4 above.

1.2.2 Payload

Any orbiting satellite contains equipment or some special component that will carry out the bulk of the mission’s agenda for success. This part is known as the payload, and it is considered among the most crucial portions of a satellite. Be it a computer that processes communications, a high-range camera for photographing the Earth, or a sensor for detecting atmospheric activity, the payload is what makes the satellite worth launching. The payload
1. SMALL SATELLITES

Figure 1.5: A P-POD containing (top) and deploying (bottom) three generic CubeSats. The P-POD’s access panels are open (top), allowing a view of the spring mechanism.

should seamlessly integrate with every aspect of the satellite and be controlled by the same systems that run everything else.

In the case of a CubeSat, the mission payload would ideally consist of very small components, such as a basic sensor and a computing/data storage device. These components would be easily portable and could fit within the specifications of the cube. In addition to sitting inside the confines of the CubeSat, the payload would also be effectively sustained by the smaller power source of the satellite. The CubeSat power source is discussed in more detail in Section 1.2.4 on page 7.

1.2.3 ANTENNA

A CubeSat will usually have one or two antennae. These are essentially long, thin metal strips designed to send and receive radio signals to and from a ground station. Ideally, these antennae will be strapped down against the satellite walls until the satellite is deployed from the P-POD, upon which they will fully unfold to their operating position. The success of a mission is usually dependent upon the satellite’s ability to receive orders and relay data back to a ground station on Earth, so it is very important that the antennae
function normally. Without communications, there is no way for the CubeSat to transfer any gathered data from the mission back to the engineers, who will analyze and interpret the data. In these cases, data is usually lost forever.

A specific example of a common antenna problem concerns antennae deployment. Finding an appropriate antennae unfolding mechanism can be an important challenge for CubeSat designers; if the antennae do not deploy correctly, the mission could be unsuccessful. My favorite approach to this problem involves wrapping an antenna around the body of the CubeSat and securing it with thin fishing wire. When the antenna is ready to be unraveled, a small current will be applied through the fishing wire; this will cause the wire to burn and break, freeing the antenna to unfold\[51\].

1.2.4 Solar Cells

Almost all small satellites will completely depend on the energy from the sun as their source of power. They spend their time in the sun charging a battery pack directly from their solar cells; they can use this stored energy when they are orbiting on the side of the Earth opposite the sun, which is dark. CubeSats are no exception to this method. Often enough, CubeSat designers employ innovative uses of solar cell technology on their satellites. For instance, a fascinating and inexpensive solution involves the use of triangular solar cells, which are produced by solar cell fabrication companies from left-over materials for their standard and more expensive rectangular cells\[†\]. While perhaps not an ideal choice for a traditional satellite’s power requirements, triangular cells are inexpensive enough that they easily help keep the satellite budget low.

There are always many challenges to overcome when installing solar cells on the small satellite. It might be difficult to find cells that will faithfully provide reasonable power throughout the duration of the planned mission. Solar cells are extremely delicate, and when exposed to the harsh temperatures and radiation of an Earth orbit, the capabilities of all but the best cells will quickly decay. It is also a challenge to find the most efficient placement of solar cells on a very small surface, such as a CubeSat’s $10 \times 10$ cm ($U \times U$) side. Real estate on a small satellite surface is not plentiful, so unorthodox cell placement strategies may need to be used\[‡\].

\[†\] Spectrolab’s Triangular Advanced Solar Cells (TASC) are a good example of this\[13\]. Normally, rectangular cells are cut out of the circular wafer of solar cell material. The TASCs are cut from the edges of the wafer that would normally go to waste, creating a considerably less expensive cell.

\[‡\] TASCs are good choices here, since they can easily be placed in irregular orientations.
1. SMALL SATELLITES

Additionally, it is also greatly encouraged that power source engineers determine the best way to add redundancy to the solar cell configuration. Simply put, it would not be acceptable to have the failure of a single cell destroy the entire solar cell system, which could occur depending on the layout of the circuitry. Therefore, it is up to an electrical engineer to design the power grid in such a way that a dead or damaged solar cell will not cause other cells to fail. The engineers must also ensure that the remaining cells will provide enough power for normal satellite operations.[34].

After having been introduced to and acquainted with the small CubeSats, we must broaden our exploration and shift our focus to the study of the physical laws that govern the behavior of all satellites, small or large; these laws are known as “orbital mechanics”. Understanding the physics of orbits can help one better comprehend the strengths of the CubeSat as a satellite, and will clarify its role in the mission scenarios described later in this paper.
CHAPTER 2

THE MECHANICS OF A MISSION

It is usually very difficult, if not impossible, for someone to immediately comprehend all of the intricacies and difficulties of a mission project. As with any satellite mission, the main aspects of the mission (purpose of the payload and the science behind the data, the way the satellite should orbit and function) are details of a complex multifaceted “problem.” As engineers, we must acquaint ourselves with the underlying problems within a mission. Additionally, dividing these down into several subproblems slowly builds a strong foundation of knowledge and makes the entire mission much more comprehensible.

When considering any satellite mission, the first things an engineer may think about are basic aspects of the mission he or she understands or must understand very well. For example, many engineers are familiar with these relevant problems concerning orbital mechanics:

- What orbit will the satellite experience?
- What other forces besides gravity are at work? Must atmospheric drag be considered?
- Will the radio antennae point to the ground station during communications?

These basic concerns are important for any satellite mission. The act of fully understanding these problems makes a good starting point towards understanding the rest of the project, including unusual subproblems that are inherently difficult to approach.

2.1 GRAVITY

The force of gravity is everywhere, always pulling matter together. It keeps our planets in orbit around the Sun, our satellites in orbit around the Earth, and our bodies from flying
2. THE MECHANICS OF A MISSION

right off the ground. It is almost always the overwhelming force in an orbital mechanics problem, and understanding the behavior of gravity in a problem is a very good first step.

While the action of gravity is now most fully described by Einstein’s General Theory of Relativity¹, there are older and simpler ways of modeling gravity and orbits. In many situations, these simpler models are sufficiently accurate for a problem while being relatively painless to solve.

2.1.1 SATELLITE MOTION AND ORBITAL ELEMENTS

When engineers design a satellite mission, one of the first things they consider is the orbit of the satellite. The altitude and orientation of the orbit will depend on how the satellite is deployed. For instance, if a CubeSat is deployed from the P-POD with a certain speed and direction, it will follow a general orbital path that can be predicted for a time.

In the early 1600s, astronomer Johannes Kepler formulated several physical laws that described the motion of planets in the sky. These laws collectively became known as Kepler’s laws of planetary motion. According to these laws, the orbit of a planet or satellite is generally not circular in shape, but rather elliptical. An ellipse is a closed curve drawn about two points, called foci.

When a satellite orbits around a larger planetary body in an elliptical orbit, the planet is positioned at one of these foci, as shown in Figure 2.1. A circular orbit occurs when the two foci essentially occupy the same point, and the orbital radius remains constant with respect to the Earth, rather than constantly changing.

There are many kinds of ellipses that the orbit of a satellite can resemble. We can describe these orbits with orbital elements. While they may take many different forms, orbital elements are always a set of six numbers that define the size and shape of the orbit and rotate it into the correct position around the Earth. Additionally, they may mark the position of a satellite onto the ellipse². We will briefly explore a common set of

---

¹The General Theory of Relativity of 1915 models gravity resulting from the curvature of space and time in the presence of mass. The curvature of space-time stems from very complex interactions between time, space, and matter.⁷
orbital elements called Keplerian elements, named after Kepler and based upon his laws of planetary motion.

1. **Semi-major axis - \( a \)**: The semi-major axis is half the length of the elliptical orbit. Figure 2.2 shows that the full length of the entire major axis is \( 2a \).

2. **Eccentricity - \( e \)**: The eccentricity is the measurement of the oblateness of the orbit, which simply describes how “squashed” the orbit is. The greater the eccentricity, the more oblate the orbit.

3. **Inclination - \( i \)**: The inclination of an orbit is the angle between the orbital plane and the equatorial plane. An orbit with a 0° inclination will orbit in the equatorial plane in the same direction as the Earth’s rotation. An orbit with an 180° inclination will revolve in the opposite direction.

4. **Longitude of the Ascending Node - \( \Omega \)**: The longitude of the ascending node is an angle between two lines pointing from the center of the Earth. Each line intersects with the surface of the Earth at a point. The first point is the vernal equinox, where the plane of the Earth’s orbit around the sun, or ecliptic plane, crosses the Earth’s equatorial plane when the sun passes from south to north (shown in Figure 2.4). The second point is called the ascending node; this is where the satellite’s orbit passes over the equator while flying from south to north. This angle is measured within the equatorial plane.

5. **Argument of Perigee - \( \omega \)**: The argument of perigee is the angle between the ascending node and the perigee (the point of
2. THE MECHANICS OF A MISSION

the satellite’s closest approach to the Earth’s surface, as shown in Figure 2.1. This angle is measured within the plane of the orbit.

6. Anomaly: Finally, the anomaly defines the position of the satellite along the orbit. The true anomaly of a satellite is the angle between the perigee of the orbit and position of the satellite, measured within the orbital plane.

The study of these orbital elements provides an excellent way to determine many interesting characteristics of orbits. For example, the orbital plane and equatorial plane always intersect at a line oriented through the center of the Earth, and a satellite close to the Earth will travel faster than when it is far away. However, simply knowing the general shape of an elliptical orbit doesn’t tell us much about gravity and how it works. In fact, Kepler’s laws do not provide any information on how to determine the orbit of any arbitrarily-placed satellite around a larger body; they simply help to characterize orbits we can already physically observe.

Orbital elements stem from Sir Isaac Newton’s Law of Universal Gravitation. Nearly one-hundred years after Kepler formulated his laws of planetary motion, Newton intently studied the physics behind gravitational phenomenon. He formulated an equation that can evaluate the gravitational force between two objects. For modern situations, we can apply the law to determine Earth’s gravitational force on a satellite, based upon the satellite’s location. Earth is essentially defined as a massive perfect sphere, or similarly as a point mass located at the Earth’s center. As we shift our focus from orbital elements to gravitational forces, we can more thoroughly explore the physics behind elementary orbital mechanics and understand how gravity affects the motion of a satellite.

Figure 2.4: The vernal equinox lies on the intersection of the ecliptic and celestial planes.
2.1 Gravity

Figure 2.5: The parameters in Newton’s universal gravitational equations.

2.1.2 Newton’s Law of Universal Gravitation

Newton’s law of Universal Gravitation for an Earth−satellite scenario gives the gravitational force between the Earth and a satellite:

\[ F_G = G \frac{M_E m_{sat}}{r^2}. \] (2.1)

The gravitational force on the satellite is the same as the force on the Earth, since they are pulling on each other with equal strength but in opposite directions (see Figure 2.5). The coefficient \( G \) is the gravitational constant, with a value of \( 6.67428 \times 10^{-11} \, \text{m}^3 \, \text{kg}^{-1} \, \text{s}^{-2} \). All Newtonian gravity for every object acts through this same constant. The mass of the Earth \( M_E \) is approximately \( 5.9742 \times 10^{24} \, \text{kg} \). The constant \( m_{sat} \) is the mass of the satellite, and \( r \) is the radial distance between the center of the Earth and satellite.

From equation 2.1, we can see that the closer a satellite is to the Earth, the greater the force. In fact, the force increases quite a bit with closer and closer distances from the center of the Earth, since it is inversely proportional to the square of \( r \). This means that if a satellite is twice as close to the Earth as before, then the force will be \( 2^2 = 4 \) times greater than before. Moving four times as close yields a gravitational force sixteen times as strong.

We have now determined that the Earth and a satellite share an attractive gravitational force. When gravitational force, or any force for that matter, is applied upon an object with mass, that object experiences acceleration. By determining the acceleration of the satellite, we can see how gravity affects its motion.

Newton’s Second Law of Motion states that the force on an object is equal to its mass times acceleration, or \( F = m a \). Therefore, since \( F_G = m_{sat} a_G \), Equation 2.1 can be rewritten
as an equation defining the acceleration of the satellite due to the Earth’s gravity:

\[ m_{sat} \vec{a}_G = -\frac{G M_E m_{sat}}{r^2} \hat{r} \]

\[ \vec{a}_G = -\frac{G M_E}{r^2} \hat{r}. \] (2.2)

For this equation, \( \vec{a}_G \) is the gravitational acceleration on the satellite, and \( \hat{r} \) is the unit vector pointing from the center of the Earth to the satellite.

Not only is the acceleration of a satellite independent of its mass, it is always pointing in the \( -\hat{r} \) direction, towards the Earth’s center. When a satellite is revolving around the Earth, it constantly feels an acceleration in that same \( \hat{r} \) direction. This is analogous to when one attaches a string to a ball and spins it around over one’s head. The tension in the string is like the gravity, and it always pulls the ball towards the center of rotation. Just as the string is constantly pulling on the ball so that it doesn’t fly away, the force of gravity is pulling on satellites so that they continuously curve around the Earth, again and again.

Describing gravity using Newton’s laws is a wonderful solution for many problems in orbital mechanics. To find an orbital path of a satellite, scientists and engineers can use either orbital elements or a simple physics simulation that constantly evaluates the acceleration of the satellite. So, not only are the laws easily solved, but the equation offers an easily comprehensible way of visualizing how the force of gravity works with simple objects. It is an ideal solution to an ideal problem.

Gravity in real life, however, is never perfectly ideal. As we prepare to leave the world of the most basic classical gravitational physics, we must realize that a satellite never perfectly follows an elliptical path. Newton’s laws of gravity are only accurate at predicting orbits for a short time into the future, until the true orbital path starts to aggressively deviate from the elliptical solution. When predictions are no longer accurate, we know that there are additional conditions at work that cause what engineers call “orbital perturbations.” These perturbative forces are a constant part of everyday life. While perturbations may occur for a variety of reasons, we can explore some of the most prevalent ones that engineers encounter.
2.1 Gravity

2.1.3 Gravitational Perturbations and Spherical Harmonics

Since the times of the ancient Greeks, the Earth’s form was known to be essentially a giant sphere. As our understanding of the Earth’s geometry improved through the years, the general shape of the Earth was found not to be perfectly spherical. Rather, it more closely resembles the form of a specific type of three-dimensional ellipse called an oblate spheroid. An oblate spheroid is an ellipsoid with equal width and depth but a smaller height, as shown in Figure 2.6.

With this knowledge of the Earth’s structure, we begin to see why the most basic form of Newton’s gravitational law is less accurate. This law assumes that the gravity of the Earth is equally distributed within a perfect sphere, not an ellipsoid. This greatly affects the long-term behavior of satellite orbits. The Earth’s gravity perturbs the satellites into taking paths that deviate from a perfect elliptical orbit. The most noticeable orbital perturbation that occurs from the oblateness of the Earth is a phenomenon known as orbital precession.

Orbital precession is the gradual rotation of an eccentric orbit. Over time, an orbital path will slowly drift; the elliptical shape of an orbital will rotate at constant or near constant rate. As a result, a satellite that followed a particular ellipse as its orbit will follow a completely different one in a few months. Therefore, describing a satellite orbit as a perfect ellipse will not be a valid method when predicting the position of the satellite far into the future. A representation of orbital precession is shown in Figure 2.7.

The study of the shape of the Earth, also known as geodesy, has made some amazing strides in recent years. By performing exhaustive data collection and analysis of the gravitational force at points around the Earth, scientists now have a more complete picture of the Earth’s structure. For example, we not only found that the Earth is roughly pear-shaped, but we now have a better idea of how mass is distributed within the planet. Instead of being uniformly dense, the Earth is quite lumpy, with pockets of more and less dense material scattered throughout its interior. The structure is very different from the ideal uniformly...
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Figure 2.8: Exaggerated representation of the true lumpiness of the Earth’s gravity, viewed from three different angles. Satellites will tend to be more attracted to high concentrations of mass.

dense sphere of matter that Newton’s gravitational law relies upon, resulting in vastly different orbital trajectories. An exaggerated representation of the Earth’s gravity can be seen in Figure 2.8.

Newton’s law of universal gravitation is a reasonably acceptable approach for predicting the orbital paths of satellites for a short time into the future. The tiny variations of the Earth’s gravity initially affect the path in virtually unnoticeable, insignificant ways. However, when attempting to propagate the orbits into the distant future, these tiny variations begin to add up over many orbits around the Earth. For example a satellite may be slightly more attracted to a massive lumpy portion of the Earth and will gradually edge toward it over several orbits. This will alter the orbital path in a very noticeable way over time. Therefore, a more detailed model of Earth’s gravity is needed when attempting to predict the trajectories of long-term orbits.

Rather than describing the gravity of the Earth as a perfect sphere or a point source, scientists and engineers can use a branch of mathematics called spherical harmonics. Spherical harmonics are a way of mapping weighted harmonic terms onto a sphere, similar to a Fourier series (mapping weighted sine and cosine terms onto a line). By simply adding a few terms of the spherical harmonics function to our original model of a spherical Earth, one can model the force of gravity due to the Earth’s oblate or pear-shaped structure. When many more terms are added, spherical harmonics can describe the complex ripples and undulations in the Earth’s gravity due to minute geographic irregularities. Additional added terms yield more accurate gravitational models and predicted orbital trajectories, at the cost of increased computing time.

Spherical harmonics are split into three categories of functions: zonal, sectoral, and tesseral harmonics. For the purposes of using spherical harmonics to model gravitational
undulations, zonal harmonics account for ripples in the latitudinal direction and sectoral harmonics model ripples in the longitudinal direction. Tesseral harmonics describe undulations in both directions.

These categories can be seen in Figure 2.9; the zonal harmonics are located in the first column, and the sectoral harmonics are located in the last column of each row. When considering zonal harmonics only, the functions do not depend upon longitude. That is, the gravity is symmetric around the Earth’s axis of rotation, and the function made purely from zonal harmonics does not change when moving around the Earth in the direction of longitude. That is, the zonal gravitational harmonics are invariant under changes in longitude.

For applications in predicting orbital trajectories, the second zonal harmonic term (highlighted in Figure 2.9) is commonly used in models. This second zonal term can define geodetic gravity instead of spherical or point-based gravity, essentially treating the Earth as if it was an ellipsoid. Adding the third zonal term will yield an even better description of Earth’s geometry, giving it a slight pear-shaped structure. Many orbital simulators use spherical harmonic terms of degree four, and some may contain up to 70 or even 300 zonal, sectoral, and tesseral terms for maximum model accuracy.

So far, we have discussed how scientists and engineers describe the motion of satellites from gravitational forces; we can see how they can account for gravitational perturbations when modeling orbits. In many ways, incorporating spherical harmonics in a model is absolutely essential for any orbital mechanics research. For many applications, irregular gravitational forces are overwhelming influences on orbiting satellites and are the primary

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1When using the second zonal harmonic term in an equation, one would multiply the term by a negative coefficient in order to add “girth” to the Earth’s equator. By using negative coefficient, the equation reverses the harmonic term’s normal behavior of pinching a sphere around its equator. This pinching behavior is demonstrated in Figure 2.9 creating a prolate ellipsoid.
causes of orbital perturbation. However, even using the best gravitational models, this
description of the Earth’s effect on satellite motion is still too ideal. We are forgetting a
very important feature of our planet, one that allows our planes to fly and makes our flags
wave. The atmosphere of the Earth, extending deep into the paths of orbiting satellites,
creates profound and unignorable perturbations in an orbit through a phenomenon known
as atmospheric drag.

2.2 Atmospheric Drag

The density of our atmosphere is sharply reduced with increases in altitude, especially when
one travels as high as the altitudes of orbiting satellites (as shown in Figure 2.10). Though
nowhere near as thick as on the surface of the Earth, high-altitude atmosphere still has
a large effect on orbiting satellites. As satellites speed through the atmosphere, they are
constantly bombarded with many atmospheric particles. As thin as this atmosphere may
be, the combined forces of each of these particles hitting the surface of a satellite add up
quickly, especially at the velocities associated with satellite orbits. Therefore, a satellite
traveling quickly through an atmosphere experiences a substantial force other than gravity.
It feels forces in the direction opposite its motion, called atmospheric drag. Drag reduces
the satellite’s energy as it orbits.

For any study including orbital mechanics, it is important to know exactly how the pres-
ence of drag will affect satellites. Scientists and engineers must determine how susceptible a
satellite is to drag forces, given its orbital path and structure. Luckily, physicists commonly
use a drag equation to determine just that. The following equation describes the force of
atmospheric drag on an object in a high-velocity scenario. Notice that the drag force is
proportional to the square of the velocity.

\[
\vec{F}_D = -\frac{1}{2} \rho v^2 C_D A \hat{v} \tag{2.3}
\]

For the drag equation, we will define \( F_D \) as the force of atmospheric drag on the satellite, \( \rho \)
as the atmospheric density, \( v \) as the satellite’s velocity through the atmosphere, \( C_D \) as the
dimensionless drag coefficient of the satellite, \( A \) as the orthogonal projectional area of the
satellite, and \( \hat{v} \) as a unit vector in the direction of the satellite’s velocity.

Equation 2.3 can be combined with Newton’s second law of motion \( F_D = m_{sat} a_D \), where
\( a_D \) is the acceleration of the satellite due to drag. We can rewrite Equation 2.3 in terms of
acceleration:

\[
\vec{a}_D = -\frac{\rho v^2 C_D A}{2 m_{sat}} \hat{v} \tag{2.4}
\]
2.2 Atmospheric Drag

Figure 2.10: Atmospheric density decreases exponentially as altitude increases\textsuperscript{[50]}. The atmospheric density scale is logarithmic, so an exponential function will appear linear. Note that the behavior is only approximately exponential for altitudes above 200 km.

Equation 2.4 shows that a satellite traveling through an atmosphere will experience an acceleration in the opposite direction of its motion, in the $-\hat{v}$ direction. Also, since $a_D$ is inversely proportional to $m_{sat}$, a heavier satellite will experience less deceleration due to drag than a lighter one of the same shape.

As previously mentioned, the intensity of drag forces depend not only on a satellite’s velocity, but also on the satellite’s structural configuration. The variable $A$, for instance, is partially determined by the structure of the satellite. The orthogonal projectional area $A$ is the area of the footprint of the satellite as it cuts through the air during flight. In more technical terms, this is the area of the satellite’s orthogonal projection onto a plane perpendicular to the direction of velocity\textsuperscript{†} shown in Figure 2.11. The larger the area, the more atmospheric particles hit the surface of the satellite as it flies through the air, which leads to a greater drag.

This situation mimics a knife slicing through the air. If the knife is swung with the blade parallel to its motion, it will encounter less drag resistance than when the flat of the blade is swung into the air. A parachute slows the descent of a skydiver by increasing the

\textsuperscript{†}This is also known as the “reference area” or, less precisely, the “cross-sectional area”\textsuperscript{[50].}
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Figure 2.11: Demonstrating the process of projecting a rotated object onto a plane. The resulting area of this projection is called the orthogonal projectional area.

possible area atmospheric particles can strike, resulting in a dramatic increase in drag. An interpretation of Equation 2.3 confirms this; we can easily see that an increase in $F_D$ is directly proportional to an increase in $A$.

The orthogonal projectional area need not stay constant throughout a satellite’s flight. It is easily altered by changing the orientation, or attitude, of the satellite with respect to its direction of velocity. Just as you can rotate a knife so that the flat of the blade moves from parallel to perpendicular with the direction of motion, a satellite can rotate in the same way. This change in attitude alters the area $A$ exposed to oncoming atmospheric particles. Conceivably, a satellite could shift from its smallest $A$ to its largest, moving through all other possible areas as it continuously changes its attitude, as shown in Figure 2.12.

Another number in Equation 2.3 that is dependent upon the structure of the satellite is the drag coefficient $C_D$. The drag coefficient is unitless, or dimensionless, so it truly is just a number. This coefficient is based upon several key structural factors of an object in question, such as the shape, surface area, and surface texture. Unfortunately, physicists usually have very little information about drag coefficients. In practice, the coefficients for conventional objects, like cars or planes, are determined empirically through wind-tunnel testing. When finding the coefficient for satellites in high orbits, uncertainty and error is inherent; the uncertainty of $C_D$ increases greatly with increasing altitude[27]. In many cases, an appropriate coefficient is selected from a table of widely-accepted values[50] for a particular satellite.

‡For low-altitude applications, the drag coefficient is usually less than or equal to 1. As a satellite’s altitude increases, the coefficient increases in turn[27], ranging anywhere around 2 to 4[50].
2.2 Atmospheric Drag

Figure 2.12: The orthogonal projectional area of a 1U CubeSat for different orientations with respect to the direction of motion. For general modeling purposes, the resulting areas can be simplified into general shapes; Area B would be equal to the area of a hexagon, and Areas A and C would equal the areas of a square.

An overwhelmingly important aspect of Equation 2.3 is $\rho$: the atmospheric density a satellite experiences. Atmospheric density is formally defined as the mass of atmospheric particles per volume of air, commonly presented in the units of kilograms per cubic meter\textsuperscript{50}. As shown in Figure 2.10, the mass density of our atmosphere varies through many orders of magnitude through various altitudes. While some sophisticated models of the atmosphere are fairly accurate at predicting atmospheric density at high altitudes\textsuperscript{31}, many extreme variations in density occur due to solar conditions, atmospheric temperature, weather patterns, and a variety of other random conditions.

Just as there are large uncertainties when determining the drag coefficient, so is equally difficult to pinpoint an exact atmospheric density at any point in the air. High-altitude densities can vary by a factor of ten or more, greatly influencing the magnitude of drag forces a satellite experiences at any given time. The popular satellite engineering and physics reference handbook \textit{Space Mission Analysis and Design} (SMAD) provides minimum, average, and maximum atmospheric densities for different altitudes\textsuperscript{50}. This provides a useful range of possible densities, acceptable for many modeling applications exploring best and worst-case drag scenarios.
2.3 Controlling Satellites in the Presence of Drag

When one views a map of all the thousands of satellites in the sky, it may look like a random mess. Satellites zip here and there, every which way, carrying out their own missions with no apparent regard for each other. That couldn’t be further from the truth; many satellites perform best when working with others. They are often required to fly together in a formation where they maintain set distances from each other in orbit. The symphony of unified and controlled movement of two or more satellites creates what are called satellite constellations. In a world with no atmosphere, we could set these satellites loose and they would fly together, like fighter jets at an air show. However, because of the pervasive perturbing effects of atmospheric drag, it is impossible to keep a constellation in a perfect formation without the use of active propulsion systems.

Satellites in a low earth orbit (LEO), an orbit with an altitude between 160 to 1000 km, are very susceptible to perturbation from drag forces. For example, the International Space Station (ISS) is currently in an extreme LEO and is always losing altitude due to energy losses from drag from the thick atmosphere. There are other kinds of frustrating perturbations; any two GPS satellites that were originally some set distance apart could potentially be far away just a short time later. This happens because drag affects each satellite in wildly different ways. The satellites could have very different masses, reference areas, or drag coefficients. There could even be a large patch of dense air that just one of the satellites happens to sail through. Whatever the reason, it is up to engineers to find ways of keeping satellites where they are supposed to be. The satellites must usually undergo corrective station-keeping by firing their thrusters at regular intervals.

It is clear that atmospheric drag must destroy the tight formation of a constellation. As mentioned before when exploring Equation \[2.4\], a heavier satellite will generally be more resistant to drag than one a few kilograms lighter. Similarly, a satellite with a smaller orthogonal projectional area will have less area to be hit by atmospheric particles, so it will also be less affected by drag. Even very tiny differences in mass or area between satellites will still change drag forces slightly. Over a long time in orbit, tiny changes can lead to very noticeable alterations in orbital behavior. The constant application of slightly different drag forces over time causes all satellites to drift away from each other. Scientists and engineers would say that constellations that drift apart are experiencing “differential drag,” or a difference in drag.

\[\text{†}\] The ISS must fire fuel-based thrusters to raise its altitude every so often. If this was not done, the entire station, including its occupants, could burn up in the atmosphere or crash to Earth.
Differential drag can be a major hindrance for satellite scientists and engineers. It becomes much more difficult to preserve a satellite’s altitude, maintain a constant position over the Earth’s surface, or keep it in synchronous movement with respect to other satellites. It can be a helpful phenomenon in some cases, though. Atmospheric drag can be used to de-orbit an old or defunct satellite, instead of leaving it to contribute to the ever-increasing quantity of dangerous space waste.

In an arguably more interesting idea, differential drag could be used to intentionally separate satellites of a constellation. So, what if there was a satellite mission that needed such a separation to be successful? What if there was a situation that called for the slow, gradual separation of two satellites throughout a lengthy mission?

This brings us to the heart of this paper. We will explore how to use differential drag as an indispensable part of an actual ongoing satellite mission: the FIREBIRD satellite project (Focused Investigations of Relativistic Electron Burst Intensity, Range, and Dynamics) of the Space Science and Engineering Laboratory (SSEL) at Montana State University in Bozeman, Montana.
CHAPTER 3

FIREBIRD

3.1 A LOOK AT THE FIREBIRD PROJECT

In a joint effort among scientists and engineers from MSU and Boston University, two identical CubeSat satellites (Figure 3.2 on page 26) will be fitted with sensors designed to detect and monitor the patterns of relativistic electron bursts (REBs) in the Earth’s magnetic field. Each satellite will be quite small, having dimensions of only $10 \times 10 \times 15$ cm (1.5U) and masses around 1.65 kg (size comparison shown in Figure 3.1). It is hoped that the mission will reveal more about the nature of REBs. Understanding these bursts is integral to understanding space weather, which influences our weather patterns on Earth. Since FIREBIRD will utilize two satellites, they can collect more data points at different locations and discern the size of individual REBs. Using two satellites ensures that FIREBIRD can observe the spatio-temporal behavior of REBs at multiple spatial scales, something a single satellite would not be able to do.

The two satellites will be placed on a “bead-on-a-string” orbit, so that one satellite will fly in front of the other. The success of this mission is dependent upon the FIREBIRD satellites slowly increasing their relative distance after P-POD deployment, separating at a speed of about a centimeter per second.

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1REBs are events of extreme high-speed electron activity. These bursts can occur when electrons interact with the Earth’s strong polar magnetic field.
By gathering data with two satellites, scientists can understand the behavior of REBs at multiple spatial scales, observing the size of pockets of REB activity and seeing if they are localized in space and time\textsuperscript{16, 41}. This task is perfectly suited for the use of Cube-Sats, since it is inexpensive to deploy two small satellites, each carrying only one sensor as payload.

3.2 Separation Strategy

Relativistic electron bursts are thought to have observable structural patterns at scales of 10 to ~100 km\textsuperscript{16, 26}, so the satellites should separate slowly enough to stay within a relative distance of around 100 km for the duration of its mission. FIREBIRD is expected to stay within this range for a minimum of four months\textsuperscript{16}. The longer the satellites stay

\textsuperscript{1}Put another way, FIREBIRD will see if these bursts generally stay in the same place and are of short duration.
within this range and observe at different spatial scales, the more data can be gathered.

Separating the FIREBIRD satellites slowly enough so that they remain within the 100 km relative distance for at least four months is not a simple task. The maneuver requires an initial separation acceleration ($\Delta v$) that must be precisely controlled to result in such low separation velocity. The FIREBIRD mission could possibly use traditional separation methods, where springs placed between the satellites would apply an initial force. However, these kinds of springs were historically only used to separate satellites from each other as fast as possible. This was done to prevent any collisions between the satellites or other difficulties. The springs are not designed to apply a precisely measured force, making them unreliable as a controlled separation mechanism\(^{26, 35}\).

Because of the unreliability associated with springs, it is a real possibility that the springs could apply too great of a separating force. This would send the satellites careening away from each other so quickly that they would sail through the 100 km separation distance limit without being able to gather enough data. When planning and designing the orbit of the FIREBIRD mission, it crucial that it is known exactly how the separation spring will affect the motion of the satellites. This means that FIREBIRD must use a mechanism that can be tuned to apply precisely the amount of force required.

Maintaining a precise orbital direction when separating is also crucial for success. If a spring applies a separation force at a nonzero angle from the direction of satellite velocity, the required “bead-on-a-string” orbit may be never realized. Instead, the FIREBIRD satellites would be slightly separated in the direction of the Earth’s radius, which would send them in to different orbits with differing orbital eccentricities. Since the satellites would no longer be following a similar orbital path, they could separate much too quickly (even with a small initial $\Delta v$).

If springs are an unreliable choice for a separation mechanism, it can be difficult to find any alternative separation mechanisms without their drawbacks. An interesting approach involves use the repulsive force of magnetism to separate FIREBIRD\(^{35}\). Magnets placed in the feet of the satellite could potentially provide a precisely-controlled initial force. However, this poses many new potential problems, one of which involves the magnets’ interaction with the Earth’s magnetic field\(^{1}\). All situations considered, perhaps the best separation mechanism must not completely rely on the application of an initial separation force from a spring or magnet.

\(^{1}\)This and several other associated issues will be discussed later in Chapter 6.
3. FIREBIRD

3.3 NOVEL APPROACHES

As hinted at the end of Chapter 2, the perturbation from atmospheric drag that dissolves the formation of constellations could help slowly separate FIREBIRD through time. Instead of using a spring or some other mechanical apparatus to apply an initial separation force, we can rely on the laws of physics to separate satellites for us. It turns out that the very small but continuous force of drag throughout an orbit might prove to be more reliable separation mechanism than a one-time, large, imprecise application of a spring force. By adjusting the otherwise identical FIREBIRD satellites’ masses, one can induce differential drag. The lighter satellite will feel significant drag accelerations from the atmosphere, and the heavier satellite will experience relatively small drag accelerations. This will cause a satellite separation over a long period of time. The constant application of different drag forces on each satellite creates a very small relative velocity which gradually increases over time. Also, since drag forces always act parallel to the direction of velocity, there is no chance of the satellites being shunted into different orbits (like they would from a spring applying an off-center force at an angle.)

Figure 3.3: Official FIREBIRD logo [16].
**Figure 3.4:** Demonstrating the relative satellite motion from Scenario 2. The letters L and H stand for “Light” and “Heavy.”
When considering the FIREBIRD mission, there are at least two scenarios that could utilize the implementation of differential drag:

1. Instead of depending on a spring whose behavior we cannot accurately predict, the separation process could rely solely upon differential drag. There is no separation via spring, simply two satellites with slightly different masses. The heavier satellite will not be affected by atmospheric drag as much as the lighter one and thus will start to lead the lighter satellite in their orbit. If the satellite mass difference is adjusted precisely, FIREBIRD will eventually drift to the edge of the maximum \( \sim 100 \text{ km} \) relative range after at least four months.

2. If an acceptable spring choice is found, the spring could give an initial separating \( \Delta v \) to the FIREBIRD satellites. The satellites would once again be constructed with slightly different masses, so that the lighter satellite would be leading the heavier satellite in orbit. Then, differential drag would gradually slow the satellite’s relative separation velocity until they eventually reach a Maximum Relative Distance (MRD) of \( \sim 100 \text{ km} \). At this point, they would begin to double back and close the distance between them, bringing them near a 0 km distance. Differential drag would continue to act at this point, separating the satellites further in the opposite direction, with the heavier satellite leading. Compared to using a spring alone, this method would theoretically increase the time the satellites stay within their MRD by about than a factor of three\(^{17}\), allowing FIREBIRD to gather much more data. A graphical representation of this scenario is shown in Figure 3.4 on page 29.

These separation strategies have not been tested\(^{26}\). According to what physicists know about drag forces, the methods will work. However, it is not readily apparent as to what exact forces and separation speeds would occur, given a mass difference between the FIREBIRD satellites. The entire problem is complex because there are many forces to consider, each of which is constantly changing. We can describe the problem in varying degrees of complexity and gain a more accurate picture of the physics behind differential drag; we can create a mathematical model.
CHAPTER 4

DERIVING THE SOLUTION

When considering a physics problem, we can create a mathematical model to aid our understanding. These models are fueled by equations that relate the motion of an object to conditions the object is experiencing. For instance, we can model satellite motion by using an equation that relates its instantaneous acceleration to its current position around the Earth, just like Equation 2.2 on page 14. Equations that form a relationship between different aspects of a satellite’s motion, such as its position, velocity, and acceleration, are called differential equations. They form the basis of all modeling in physics.

4.1 Differential Equations

A differential equation defines a relationship between the value of some function and the value of one or more of that function’s derivatives (rate of change). Differential equations themselves are not functions, but they can describe the general behavior of a function regardless of whether that function is known. The act of solving the differential equation yields a function as a possible solution.

Consider a simple physics scenario involving a freely falling object with mass $m$. Close to the surface of the Earth, objects falling under gravity accelerate at a rate of $g = -9.8 \text{ m}\cdot\text{s}^{-2}$ (an acceleration in the downward and negative direction). All objects near the Earth’s surface fall at the same acceleration $g$, regardless of their mass.† Since acceleration $a$ is the second derivative ($\ddot{x}$) of the object’s position ($x$) with respect to time,‡ we can express a falling object’s motion with a differential equation:

$$\ddot{x} = g.$$

†This can be found through Equation 2.2 by setting $r$ equal to the radius of Earth.
‡The first derivative of $x$ (the rate of change of $x$ with respect to time) is written as $\dot{x}$. The second derivative of $x$ (the rate of change of the first derivative $\dot{x}$ with respect to time) is written as $\ddot{x}$. 

31
When we define \( \ddot{x} = g \), our differential equation says only one thing about the object’s motion: the acceleration \( \ddot{x} \) of a falling object is always equal to \( g \). Our ultimate aim is to solve the differential equation, so we can determine the position \( x \) of the object as a function of time \( t \). Since our acceleration is simply a constant \( g \), we can easily find a function that satisfies this differential equation by finding its first and second anti-derivatives:

\[
\ddot{x} = g = -9.8 \\
\dot{x} = gt + v_0 = -9.8t + v_0 \\
x = \frac{1}{2}gt^2 + v_0t + x_0 = -4.9t^2 + v_0t + x_0.
\]

Here, \( v_0 \) is the falling object’s initial velocity in the vertical direction, and \( x_0 \) is the object’s initial position above the ground. These initial inputs of velocity and position are known as initial conditions, and they describe the object’s “beginning state.” The solution of the differential equation can take many different forms, depending on the initial conditions. By setting an initial velocity \( v_0 \) and position \( x_0 \) for the object, we can track the position of the object \( x \) at any future point in time \( t \). We now know the position of a falling object as a function of time.

During the course of this paper, we have already come across several other differential equations. In Equation 2.2 on page 14, we described the gravitational acceleration \( a_G \) on an object as being dependent upon its radial distance \( r \) from the center of the Earth. Below, we rewrite that equation; for simplicity, we will define \( \mu \) as the constant \( G \cdot M_E \).

\[
\vec{a}_G = -\frac{\mu}{r^2} \hat{r}
\]

We can rewrite Equation 2.2 in the more familiar form of a differential equation. In its simplest form, we can create a one-dimensional differential equation describing the radial gravitational acceleration of an object (\( \ddot{x} \)) given its radial position \( x \):

\[
\ddot{x} = -\frac{\mu}{x^2}.
\]

This differential equation is more complex than our previous one; the second derivative is no longer a constant, but inversely proportional to the square of the position. It cannot be solved through such simple methods as the previous equation, and it is not straightforward to obtain a function that will give us the radial position \( x \) at any future point in time. However, all is not lost. When extending this differential equation to describe motion in two and three spatial dimensions, one can obtain a clear analytical (or closed-form) solution to this differential equation: an elliptical orbit described by Keplerian orbital elements.
So, it is possible to clearly describe the motion of an object experiencing simple gravity as a function of time, noting that the object will model an unchanging elliptical orbit. What would happen if we added orbital perturbations to the mix? Consider an object experiencing both gravitational and drag forces, relying partly on the expression for $a_D$ in Equation 2.4 on page 18. What results is a differential equation complex enough that its exact solution cannot be expressed analytically. It is impossible to derive an analytical function that could account for the many possible behaviors the differential equation describes. The trajectory of an orbit perturbed by drag forces cannot be easily described by an exact function. In truth, very few differential equations have solutions that can be fully expressed by an exact function.

4.2 Numerical Methods

When solving differential equations, there are two main approaches. Either an exact analytical solution is found, or the equations may be solved numerically. Instead of keeping the equation in terms of symbolic variables, specific numbers are substituted in for variables and constants, hence a “numerical” approach.

Ideally, solving differential equations with a numerical method would yield exactly the same result as an analytical solution. However, one arrives at numerical solution by approximating the behavior of the derivatives, rather than determining exact behavior. The values of the function and its derivatives are represented over a series of discrete steps, rather than with respect to a continuous variable (like with an analytical solution). Representing the behavior of a differential equation over steps lets one approximate its derivative terms via the finite difference method.

Finite differences approximate derivatives of a function by using the value of the function at select points. As we explore this idea, consider a function $f(t)$ that changes with time. By definition of a derivative, the first derivative of a function $f(t)$ at $t = a$ is

$$
\dot{f}(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t}.
$$

Therefore, an approximation for that derivative could appropriately be

$$
\dot{f}(a) \approx \frac{f(a + \Delta t) - f(a)}{\Delta t} \text{ for small values of } \Delta t.
$$

There is, of course, some error associated with this method; we are attempting to determine the derivative at $t = a$ by evaluating the function at a couple of discrete steps ($a$ and $a + \Delta t$) rather than continuously. We cannot account for how $\dot{f}$ is influenced by
4. DERIVING THE SOLUTION

anything between \( f(a) \) and \( f(a + \Delta t) \). However, as smaller \( \Delta t \) values are used in a finite difference, the approximation becomes increasingly closer to the actual value of the derivative.

There are many forms that finite difference approximations of derivatives can take, and some converge to the actual derivative more quickly than others. A so-called “first-order” method, such as the one we wrote for the first derivative \( \dot{f} \), has a numerical error directly proportional to \( \Delta t \) when the value of \( \Delta t \) is small, while the error of a second-order method\[^1\] is generally proportional to \( \Delta t^2 \). This means that for a first-order method, a decrease in \( \Delta t \) would proportionally reduce the error when \( \Delta t \) is small; for a second-order method, a reduction in \( \Delta t \) by half would generally reduce the error by a factor of four. Higher-order finite difference methods have increasingly better results; common methods for scientific applications range from fourth-order upward.

How can one use the finite-difference method to fully solve the differential equation? Let’s say that we had a first-order differential equation that describes the relationship between some function \( f(t) \) and its derivative \( \dot{f}(t) \). We could also have some initial conditions where \( t = a \), so we know the value for \( f(a) \). However, this is all we know about the function; we currently don’t know the value for \( f(t) \) a little bit into the future, at \( t = a + \Delta t \). We could solve for the value of \( f(a + \Delta t) \) by using the finite difference method explored earlier.

\[
\frac{f(a + \Delta t) - f(a)}{\Delta t} \approx \dot{f}(a)
\]
\[
f(a + \Delta t) - f(a) \approx \dot{f}(a) \Delta t
\]
\[
f(a + \Delta t) \approx f(a) + \dot{f}(a) \Delta t
\]

Since we have a numeric value for \( f(a) \), we can substitute that into our equation. Also, we know that \( \dot{f}(a) \) is in terms of \( f(a) \), so we can calculate a numeric value for that, too. Finally, we wish to look forward in time by some chosen number \( \Delta t \). We now have a specific way of finding the numeric value of \( f(t) \) at a time \( \Delta t \) from where we originally started! We could then repeat this process again, using the value of \( f(a + \Delta t) \) and solving for \( f(a + 2 \Delta t) \), and so on.

As one might realize, we are determining the values of \( f(t) \) by moving forward in successive steps of \( \Delta t \) through time. After evaluating the equation at each “time-step,” we feed the results into the equation for the next time-step. This type of recursive equation

\[^1\]A higher-order method for finding a derivative at point \( a \) would use values of the function at more than one step around \( a \). Utilizing more points means that the finite difference uses more information about the function, which leads to more accurate approximations of its derivative.
is called a difference equation because it alters the value of the function at each step by adding on successive differences, rather than having the function change continuously due to derivatives. The smaller we make the “step-size” of time with each step, the smaller the differences added and the more accurate the solution.

No numerical method gives a perfect solution to a differential equation, because we must move ahead in discrete steps. If the step-size is large enough, significant errors can be introduced into the solution. Numerical methods assume that the function’s behavior between time-steps follows a predictable pattern; a first-order method makes the assumption that the curve of the function will be linear between steps. In actuality, the curve of the function might be very different, causing one to overestimate or underestimate the value of $f(t)$ at each step. This will yield successively inaccurate estimates as error accumulates over many steps, making the solution practically useless.

If the step-size is reduced, $f(t)$ has less of a chance to drastically change between successive steps. This might mean that the true behavior of $f(t)$ is more predictable and easily approximated by the numerical method, which leads to greater accuracy. The flip-side is that since we are moving forward in time by a smaller amount at each step, more steps must be computed to reveal the solution to a desired time. If it were somehow possible to reduce the step-size down to zero, the solution would be exact provided that infinitely many steps were calculated.

Let’s apply this idea to a real-life problem: determining a satellite’s orbital path around the Earth. The initial conditions for the satellite would define its initial speed and direction in space, as well as its initial position with respect to the Earth. Since a satellite’s acceleration is dependent on its position (in the simplest case, the position is some distance $r$ from the Earth’s center), we can determine the initial instantaneous acceleration of the satellite. This means that we can predict its velocity at the next step, and from the velocity we can find its next position. Then, we can use these new values of position and velocity in the same way to calculate a new acceleration, which leads to the position and velocity at the next step. This process is repeated over and over again, until our solution has been determined through enough steps of $\Delta t$. Here, enough “simulation time” has passed to see a satisfactory solution curve.

Since the majority of differential equations do not have an analytical solution, numerical computation is a powerful choice; the method is limited only by a computer’s ability to perform calculations at many time steps with a decently small step-size. Then, any differential equation with initial conditions can be solved.
4. DERIVING THE SOLUTION

4.3 EQUATIONS OF THE MODEL

To create equations defining the motion of an orbiting satellite, we must describe its motion by three differential equations: one for each of our three dimensions of space. The equations can account for gravity, drag, and any other phenomenon we may include. We can start from basic assumptions about the physics of orbital mechanics, and slowly compile the ways the satellite’s acceleration could be changed with respect to position and velocity. When we finish, we will have a system of differential equations.

4.3.1 SIMPLE POINT GRAVITY

For any orbital dynamics problem, gravity is considered the overwhelming force. To fully understand how to create equations of motion describing both simple and more complex situations, it is best we touch briefly on the concept of gravitational potential, and how it relates to the more familiar Newton’s equations of motion. The gravitational potential at some distance \( r \) from the center of the Earth is

\[
V = -\frac{\mu}{r}. \tag{4.1}
\]

Once again, we are using the constant \( \mu \) for \( G \cdot M_E \). The gravitational potential can be visualized as a sort of “gravitational field.” The potential \( V \) is the amount of energy per unit of mass that is influenced by the gravitational field (Joules per kilogram in SI units). At a distance \( r = \infty \), the potential is zero, and the field is not acting on any objects.

We can relate the potential \( V \) to the gravitational acceleration \( \vec{a} \) on a satellite; both are independent of satellite mass. We simply find the negative gradient of \( V \), since \( \vec{a} = -\nabla V \).

The gradient \( \nabla \) of a function is a vector field with vectors pointing in the direction of the greatest increase of the function\[37\]; essentially, it is a “vector” derivative made up of three partial derivatives in three directions. In spherical coordinates, where \( \phi \) is in the direction of longitude and \( \theta \) is in the direction of latitude measured from the equator, the negative gradient\[37\] of the gravitational potential \( V \) is

\[
-\nabla V = -\left( \frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \frac{1}{r \cos \theta} \frac{\partial V}{\partial \phi} \hat{\phi} \right) \tag{4.2}
\]

\[
= -\frac{\mu}{r^2} \hat{r} + 0 \hat{\theta} + 0 \hat{\phi}
\]

\[
\vec{a}_G = -\frac{\mu}{r^2} \hat{r}.
\]

The result is an equation defining the acceleration of a satellite given \( r \), its radial distance from the center of the Earth. Note that there are no changes in the potential in either the
4.3 Equations of the Model

latitudinal or longitudinal directions; this makes these partial derivatives simply zero. Also, we can see this equation is none other than Equation 2.2.

We can describe the acceleration of a satellite more conveniently in three separate equations in Cartesian coordinates, by recognizing that the direction of the satellite’s acceleration is pointing in the \(-\hat{r}\) direction.

We begin with our equation for acceleration, noting that \(r\) is the magnitude of the radial position vector \(\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}\), so that \(r \equiv \sqrt{x^2 + y^2 + z^2}\).

\[
\vec{a}_G = -\frac{\mu}{r^2} \hat{r}
\]

We then rewrite the unit vector \(\hat{r}\) as the position vector \(\vec{r}\) divided by its magnitude:

\[
\vec{a}_G = -\frac{\mu}{r^2} \left( \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} \right) = -\frac{\mu}{r^2} \left( \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} \right) \cdot r.
\]

We can split Equation 4.3 into three separate equations describing the acceleration in the \(x\), \(y\), and \(z\) directions independently:

\[
a_{Gx} = -\frac{\mu}{r^2} \frac{x}{r}, \quad a_{Gy} = -\frac{\mu}{r^2} \frac{y}{r}, \quad a_{Gz} = -\frac{\mu}{r^2} \frac{z}{r}.
\]

These are equations of motion for simple Newtonian gravity for a spherical Earth. Since they relate the position of a satellite to its acceleration, they are inherently “differential equations.” So, Equation 4.4 can be alternatively written as

\[
\ddot{x} = -\frac{\mu}{r^2} \frac{x}{r}, \quad \ddot{y} = -\frac{\mu}{r^2} \frac{y}{r}, \quad \ddot{z} = -\frac{\mu}{r^2} \frac{z}{r}.
\]

\[\text{†}\]

Cartesian geometry is three-dimensional, with \(x\), \(y\) and \(z\) coordinates from the origin (in this case, the center of the Earth). The axes of these coordinates point in three perpendicular directions, represented by the unit vectors \(\hat{i}, \hat{j}\) and \(\hat{k}\) (corresponding to \(x\), \(y\), and \(z\) respectively).
4.3.2 SPHERICAL HARMONICS OF THE GRAVITATIONAL POTENTIAL

Delving any further into gravitational physics requires a bit more effort than simply rearranging existing equations. Once again, we will be describing gravity with potential $V$, as if it were a field. To increase the realism of our model’s gravitational field, we must incorporate equations that describe gravitational perturbations from a non-spherical Earth. To do this, we shall add extra terms of spherical harmonics of the Earth’s potential to Equation 4.1. Each harmonic term will add more complex structure to our description of Earth’s gravity. The largest term will account for the Earth’s natural oblateness, and any successive term shall introduce extra mounds, bumps, and ripples.
Spherical harmonics of the potential are commonly organized by two identifying numbers: a degree of $\ell$ and an order of $m$, where $\ell$ and $m$ are positive integers and $m \leq \ell$. For our purposes, we can describe a spherical harmonic term of our potential with the function $V_{\ell,m}(r, \phi, \theta)$. By themselves, each term contributes a tiny portion of complexity and detail to the full gravitational potential $V$. The strength by which each term contributes is governed by the signs and magnitudes of the terms’ spherical harmonic coefficients. If it were possible to sum an infinite number of potential harmonic terms, we could have a complete model of Earth’s gravity in perfect resolution.

For practical purposes, it is usually not necessary to include very many terms in such a summation; the first few potential harmonic terms contribute the most to the full gravitational potential. The $V_{0,0}$ term does not change the gravitational potential described in Equation 4.1 and would simply be the potential for a spherical Earth. Harmonics of the lowest degrees and orders describe large, over-arching structural details. Theoretically, to add the largest of these alterations, we can take our original potential and add on the term $V_{1,0}$:

$$V = -\frac{\mu}{r} + V_{1,0}.$$ 

Since $V_{1,0}$ is a potential harmonic term where $\ell = 1$ and $m = 0$, we can look at Figure 4.1 to see what effect it will have on the gravitational potential. The harmonic associated with $V_{1,0}$ could add a bulge to the top of our previously spherically-symmetrical potential, while subtracting an equal amount at the bottom. This would make the resultant potential top-heavy, so a satellite at a distance $r$ above the Earth would experience a stronger gravitational potential than if it were at a distance $r$ below. If our term contained a negative harmonic coefficient, a bulge would appear at the bottom instead, and the effect on the potential would be reversed.

When describing the gravitational potential of the Earth, one important assumption is made: the origin of the coordinate system lies at the Earth’s center-of-mass. Since the addition of a $V_{1,0}$ term to the potential adds a bulge to the top or bottom, this would imply that the Earth was top or bottom-heavy, and the Earth’s center-of-mass would be shifted above or below the origin. This breaks our assumption that the center-of-mass and the origin lie at the same point. By convention, to preserve this assumption, we set the associated harmonic coefficient of the $V_{1,0}$ term equal to zero; it will not contribute to the potential $V$. Likewise, the $V_{1,1}$ term will also not contribute. Instead, for the first potential harmonic term, we skip directly to $V_{2,0}$. 
4. DERIVING THE SOLUTION

Adding the potential harmonic term $V_{2,0}$ effectively controls the amount of bulge around the Earth’s equator. By introducing this term with a negative harmonic coefficient, the gravitational potential will become more bulbous; a positive coefficient would pinch the potential inward around the equator, as shown in Figure 4.1 for $\ell = 2$ and $m = 0$. The coefficient for $V_{2,0}$ is the largest in magnitude of any of the potential harmonic terms $V_{\ell,m}$, and is therefore the largest contributor to the total potential $V$, treating the Earth as an oblate object.

As mentioned before in Section 2.1.3 on page 17, spherical harmonics are split into zonal, tesseral, and sectoral harmonics. The zonals consist of all harmonics with an order $m$ of zero; this corresponds to the first column in Figure 4.1. As previously explained, zonal harmonics are invariant under changes in longitude. Therefore, if one wishes to create a model of the gravitational potential dependent only on radial distance and latitude, one could implement only zonal $V_{\ell,0}$ terms. This would greatly simplify the model by ignoring the effects of the Earth’s rotation on the gravitational potential, since the zonal harmonics are symmetric about the Earth’s axis of rotation. For most practical uses, the first few zonal harmonics provide sufficient accuracy for long-term orbital propagations\(^5\).

Spherical harmonics are the solutions of Laplace’s equation, $\nabla^2 f = 0$. This partial differential equation is the basis of the study of potential theory\(^5\) \(^48\). The Laplacian $\nabla^2$ of the potential $V$ is equal to $4\pi G \rho$, which is a variation of Poisson’s equation using Gauss’ law of gravity\(^10\) ($\rho$ is the mass density of a gravitational body)\(^7\). Laplace’s equation is a special case of Poisson’s equation. The solutions of Poisson’s equation for the potential $V$ give us the spherical harmonics of the potential:

$$V(r, \phi, \theta) = \sum_{\ell, m=0}^{\infty} V_{\ell,m}(r, \phi, \theta) = -\frac{\mu}{r} \sum_{\ell=0}^{\infty} \left(\frac{a_E}{r}\right) \ell \sum_{m=0}^{\infty} P_{\ell,m}(\sin \theta) \left[ C_{\ell,m} \cos m\phi + S_{\ell,m} \sin m\phi \right],$$  \hspace{1cm} (4.6)

where $a_E$ is the semi-major axis of the elliptical Earth; $C_{\ell,m}$ and $S_{\ell,m}$ are spherical harmonic coefficients; $r$, $\phi$, and $\theta$ are respectively the radial distance from the Earth’s center, the longitude, and the latitude in an Earth-fixed coordinate system\(^4\); and $P_{\ell,m}$ are the

---

\(^1\)Gauss’ law of gravity is a physically equivalent way of stating Newton’s law, but in the language of vector calculus.

\(^2\)This coordinate system rotates in space along with the Earth; a coordinate with certain longitude $\phi$ will always represent the same location on the Earth, as opposed to an inertial coordinate system. However, even though we are working with a non-inertial coordinate system, we will find on page 42 that the validity of our equations of motion will not be nullified by using harmonics that are exclusively zonal.
4.3 Equations of the Model

Associated Legendre Functions. The spherical harmonic coefficients and the Associated Legendre Functions are of degree \( \ell \) and order \( m \)

This expression for the potential \( V \) is dependent upon both longitude \( \phi \) and latitude \( \theta \), because it includes spherical harmonic terms of degree \( \ell \) and order \( m \). If we wish to only use zonal harmonic terms so our model is only dependent on the angular direction of latitude, we shall set the order \( m \) to be zero.

Setting \( m = 0 \) lets Equation 4.6 simplify to only include \( V_{\ell,0} \) terms to describe a rotationally-symmetric gravitational potential:

\[
V(r, \theta) = \sum_{\ell=0}^{\infty} V_{\ell,0}
\]

\[
= -\frac{\mu}{r} \sum_{\ell=0}^{\infty} \left( \frac{aE}{r} \right)^\ell P_{\ell,0}(\sin \theta) C_{\ell,0}.
\]  \hspace{1cm} (4.7)

For the purposes of the model, we will add the first two relevant zonal potential harmonics \( V_{2,0} \) and \( V_{3,0} \) to our original spherical potential, seen in Equation 4.8 below. By convention, the first zonal harmonic term \( V_{1,0} \) contributes zero to the final potential \( V \), assuming the center of the coordinate system coincides with the center of Earth’s mass. The second harmonic term describes how the Earth’s eccentricity changes the gravitational field; the third harmonic term has the field reflect that of a pear-shaped Earth.

\[
V = -\frac{\mu}{r} - \frac{\mu}{r} \left( \frac{aE}{r} \right)^2 P_{2,0}(\sin \theta) C_{2,0} - \frac{\mu}{r} \left( \frac{aE}{r} \right)^3 P_{3,0}(\sin \theta) C_{3,0}.
\]  \hspace{1cm} (4.8)

To find the functions \( P_{2,0} \) and \( P_{3,0} \), we shall calculate Legendre Polynomials using Rodrigues’ formula, shown in Equation 4.9. The Legendre Polynomials are a special form of the Associated Legendre Functions for when \( m = 0 \).

\[
P_{\ell,0}(g) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dg^\ell} \left( g^2 - 1 \right)^\ell.
\]  \hspace{1cm} (4.9)

Evaluating Equation 4.9 for \( \ell = 2 \) and \( \ell = 3 \) gives us

\[
P_{2,0}(g) = \frac{3}{2} g^2 - \frac{1}{2} \quad \text{and} \quad P_{3,0}(g) = g \left( \frac{5}{2} g^2 - \frac{3}{2} \right).
\]

Substituting our determined Legendre Polynomials back into Equation 4.8 gives

\[
V = -\frac{\mu}{r} \left[ \frac{\mu}{r} \left( \frac{aE}{r} \right)^2 \left( \frac{3}{2} \sin^2 \theta - \frac{1}{2} \right) C_{2,0} \right] - \left[ \frac{\mu}{r} \left( \frac{aE}{r} \right)^3 \sin \theta \left( \frac{5}{2} \sin^2 \theta - \frac{3}{2} \right) C_{3,0} \right].
\]  \hspace{1cm} (4.10)
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Now, just as we did in Equation 4.12, we will compute the acceleration by calculating the negative gradient of the gravitational potential in Earth-fixed spherical coordinates:

\[
-\nabla V = -\frac{\partial V}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} - \frac{1}{r \cos \theta} \frac{\partial V}{\partial \phi} \hat{\phi}
\]  
\[\text{(4.11)}\]

\[
\mathbf{a_G} = \left( \frac{\mu}{r^2} - \left[ \frac{\mu}{r^2} \left( \frac{a_E}{r} \right)^2 \left( \frac{9}{2} \sin^2 \theta - \frac{3}{2} \right) C_{2,0} \right] - \left[ \frac{\mu}{r^2} \left( \frac{a_E}{r} \right)^3 \sin \theta \left( 10 \sin^2 \theta - 6 \right) C_{3,0} \right] \right) \hat{r} 
+ \left( \frac{\mu}{r^2} \left( \frac{a_E}{r} \right)^2 \left( 3 \cos \theta \sin \theta \right) C_{2,0} \right) + \left[ \frac{\mu}{r^2} \left( \frac{a_E}{r} \right)^3 \left( \frac{15}{2} \sin^2 \theta - \frac{3}{2} \right) C_{3,0} \cos \theta \right] \hat{\phi}.
\]

Note that the \( \phi \) component of acceleration is necessarily zero, since the zonal harmonics are invariant under changes in longitude.

Using the transformation matrix operation shown in Equation 4.12, we can now begin the conversion process from spherical coordinates to the more familiar Cartesian coordinate system by rotating the spherical unit vectors \( \hat{r}, \hat{\theta} \) and \( \hat{\phi} \) to the Cartesian unit vectors \( \hat{x}, \hat{y}, \) and \( \hat{z} \).

\[
\begin{bmatrix}
    a_{G_x} \\
    a_{G_y} \\
    a_{G_z}
\end{bmatrix}
= 
\begin{bmatrix}
    \cos \theta \cos \phi & -\sin \theta \cos \phi & -\sin \phi \\
    \cos \theta \sin \phi & -\sin \theta \sin \phi & \cos \phi \\
    \sin \theta & \cos \theta & 0
\end{bmatrix}
\begin{bmatrix}
    a_{G_r} \\
    a_{G_{\theta}} \\
    a_{G_{\phi}}
\end{bmatrix}
\]
\[\text{(4.12)}\]

Applying this transformation matrix of Equation 4.12 to our vector \( \mathbf{a_G} \) and simplifying yields a system of acceleration equations in Cartesian coordinates with spherical coordinate variables.

\[
\begin{align*}
a_{G_x} &= -\frac{\mu}{r^2} \left[ 1 + \left( \frac{a_E}{r} \right)^2 C_{2,0} \left( \frac{15}{2} \sin^2 \theta - \frac{3}{2} \right) + \left( \frac{a_E}{r} \right)^3 C_{3,0} \left( \frac{35}{2} \sin^3 \theta - \frac{15}{2} \sin \theta \right) \right] \cos \theta \cos \phi \\
\end{align*}
\]

\[
\begin{align*}
a_{G_y} &= -\frac{\mu}{r^2} \left[ 1 + \left( \frac{a_E}{r} \right)^2 C_{2,0} \left( \frac{15}{2} \sin^2 \theta - \frac{3}{2} \right) + \left( \frac{a_E}{r} \right)^3 C_{3,0} \left( \frac{35}{2} \sin^3 \theta - \frac{15}{2} \sin \theta \right) \right] \cos \theta \sin \phi \\
\end{align*}
\]

\[
\begin{align*}
a_{G_z} &= -\frac{\mu}{r^2} \left[ \sin \theta + \left( \frac{a_E}{r} \right)^2 C_{2,0} \left( \frac{15}{2} \sin^3 \theta - \frac{9}{2} \sin \theta \right) + \left( \frac{a_E}{r} \right)^3 C_{3,0} \left( \frac{35}{2} \sin^4 \theta - 15 \sin^2 \theta + \frac{3}{2} \right) \right]
\end{align*}
\]

Since the zonal harmonics are invariant under changes in longitude, they are also unchanged by the rotation of the Earth. If we were evaluating spherical harmonics for \( m < 0 \), then it would be necessary to rotate our coordinate system from an Earth-fixed to an inertial coordinate system. However, we are only concerned with the zonal harmonics, which are completely symmetric about the Earth’s axis of rotation, so this coordinate conversion is not mathematically necessary. At any rate, we can now proceed as if we were working in Earth-inertial Cartesian coordinates.

To complete this coordinate conversion, we must change our \( \phi \) and \( \theta \) variables into their equivalent \( x, y, z \) variables. We can use the following equivalences for \( \phi \) and \( \theta \)\[52\], if the
4.3 Equations of the Model

function “arctan \( \frac{w}{v} \)” is defined as the counter-clockwise angle from the vector \( \langle 1, 0 \rangle \) to the vector \( \langle v, w \rangle \) in all quadrants of a 2D Cartesian plane:

\[
\phi = \arctan \left( \frac{y}{x} \right) \quad \text{and} \quad \theta = \arctan \left( \frac{z}{\sqrt{x^2 + y^2}} \right).
\]

Note that we have four trigonometric functions in our acceleration equations: \( \sin \theta \), \( \cos \theta \), \( \sin \phi \), and \( \cos \phi \). Substituting our identities for \( \phi \) and \( \theta \) into these trigonometric functions and simplifying, keeping in mind that the arctan function is sufficiently defined for all quadrants, gives us:

\[
\sin \theta = \sin \left( \arctan \left( \frac{z}{\sqrt{x^2 + y^2}} \right) \right) = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\
\cos \theta = \cos \left( \arctan \left( \frac{z}{\sqrt{x^2 + y^2}} \right) \right) = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \\
\sin \phi = \sin \left( \arctan \left( \frac{y}{x} \right) \right) = \frac{y}{\sqrt{x^2 + y^2}} \quad \text{and} \\
\cos \phi = \cos \left( \arctan \left( \frac{y}{x} \right) \right) = \frac{x}{\sqrt{x^2 + y^2}}.
\]

We can see that our radial variable \( r \) sits in these equations: \( \sqrt{x^2 + y^2 + z^2} \). We shall replace this term with \( r \), and the smaller radial term we can denote \( \sqrt{x^2 + y^2} = r_0 \).

\[
\sin \theta = \frac{z}{r}, \quad \cos \theta = \frac{r_0}{r}, \quad \sin \phi = \frac{y}{r_0}, \quad \cos \phi = \frac{x}{r_0}
\]

We can now begin to substitute in these values for our trigonometric functions.

\[
a_{Gx} = -\frac{\mu}{r^2} \left[ 1 + \left( \frac{aE}{r} \right)^2 C_{2,0} \left( \frac{15}{2} \left( \frac{z}{r} \right)^2 - \frac{3}{2} \right) + \left( \frac{aE}{r} \right)^3 C_{3,0} \left( \frac{35}{2} \left( \frac{z}{r} \right)^3 - \frac{15 z}{2 r} \right) \right] \frac{r_0}{r} \frac{x}{r_0} \\
a_{Gy} = -\frac{\mu}{r^2} \left[ 1 + \left( \frac{aE}{r} \right)^2 C_{2,0} \left( \frac{15}{2} \left( \frac{z}{r} \right)^2 - \frac{3}{2} \right) + \left( \frac{aE}{r} \right)^3 C_{3,0} \left( \frac{35}{2} \left( \frac{z}{r} \right)^3 - \frac{15 z}{2 r} \right) \right] \frac{r_0}{r} \frac{y}{r_0} \\
a_{Gz} = -\frac{\mu}{r^2} \left[ \frac{z}{r} + \left( \frac{aE}{r} \right)^2 C_{2,0} \left( \frac{15}{2} \left( \frac{z}{r} \right)^3 - \frac{9 z}{2 r^2} \right) + \left( \frac{aE}{r} \right)^3 C_{3,0} \left( \frac{35}{2} \left( \frac{z}{r} \right)^4 - \frac{15 z}{2 r^2} + \frac{3}{2} \right) \right] \frac{r_0}{r} \frac{z}{r_0}
\]

From this, it is evident that the \( r_0 \) variables cancel out entirely. Further rearrangement reveals the separation of the individual harmonic terms, as shown in Equation 4.13. Note that the first term of each acceleration component is simply the gravitational acceleration.
due to a uniform point-mass, as previously seen in Equation 4.4:

\[ a_{G_x} = -\mu \frac{x}{r^2} - \left[ \frac{\mu}{r^2} \left( \frac{a_E}{r} \right)^2 C_{2,0} \left( \frac{15}{2} \left( \frac{z}{r} \right)^2 - \frac{3}{2} \right) \frac{x}{r} \right] - \left[ \frac{\mu}{r^2} \left( \frac{a_E}{r} \right)^3 C_{3,0} \left( \frac{35}{2} \left( \frac{z}{r} \right)^3 - \frac{15}{2} \left( \frac{z}{r} \right) \right) \frac{x}{r} \right] \]

\[ a_{G_y} = -\mu \frac{y}{r^2} - \left[ \frac{\mu}{r^2} \left( \frac{a_E}{r} \right)^2 C_{2,0} \left( \frac{15}{2} \left( \frac{z}{r} \right)^2 - \frac{3}{2} \right) \frac{y}{r} \right] - \left[ \frac{\mu}{r^2} \left( \frac{a_E}{r} \right)^3 C_{3,0} \left( \frac{35}{2} \left( \frac{z}{r} \right)^3 - \frac{15}{2} \left( \frac{z}{r} \right) \right) \frac{y}{r} \right] \]

\[ a_{G_z} = -\mu \frac{z}{r^2} - \left[ \frac{\mu}{r^2} \left( \frac{a_E}{r} \right)^2 C_{2,0} \left( \frac{15}{2} \left( \frac{z}{r} \right)^2 - \frac{3}{2} \right) \frac{z}{r} \right] - \left[ \frac{\mu}{r^2} \left( \frac{a_E}{r} \right)^3 C_{3,0} \left( \frac{35}{2} \left( \frac{z}{r} \right)^3 - \frac{15}{2} \left( \frac{z}{r} \right) \right) \frac{z}{r} \right] \]

\[ (4.13) \]

4.3.3 Drag

We have just created a model that describes the acceleration on the FIREBIRD CubeSats due to gravity alone. Now, since our problem requires that we know the acceleration due to atmospheric drag, we should include this in our model. Recall the expression for \( a_d \) in Equation 2.4 on page 18. Here, we have an equation that gives us the acceleration due to drag on an object given the object’s velocity \( v \). This acceleration acts in the \( -\hat{v} \) direction.

We can rewrite Equation 2.4 in Cartesian coordinates. Note that \( v \) is the magnitude of the object’s velocity vector \( \vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \), so that \( v = \sqrt{v_x^2 + v_y^2 + v_z^2} \).

\[
\vec{a}_D = -\rho \frac{v^2 C_D A}{2 m_{sat}} \vec{v} = -\rho \frac{v^2 C_D A}{2 m_{sat}} \left( \frac{v_x}{v} \hat{i} + \frac{v_y}{v} \hat{j} + \frac{v_z}{v} \hat{k} \right) = -\rho \frac{v^2 C_D A}{2 m_{sat}} \left( v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \right) = -\rho \frac{v C_D A}{2 m_{sat}} v \left( v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \right)
\]

\[ (4.14) \]

We can split Equation 4.14 into three separate equations describing the acceleration in the x, y, and z directions independently:

\[ a_{Dx} = -\rho \frac{v C_D A}{2 m_{sat}} v_x, \quad a_{Dy} = -\rho \frac{v C_D A}{2 m_{sat}} v_y, \quad a_{Dz} = -\rho \frac{v C_D A}{2 m_{sat}} v_z. \]

\[ (4.15) \]

Writing Equation 4.15 in terms of x derivatives gives us the familiar differential equation form:

\[ \ddot{x} = -\rho \frac{v C_D A}{2 m_{sat}} \dot{x}, \quad \ddot{y} = -\rho \frac{v C_D A}{2 m_{sat}} \dot{y}, \quad \ddot{z} = -\rho \frac{v C_D A}{2 m_{sat}} \dot{z}. \]

\[ (4.16) \]
4.3 Equations of the Model

4.3.4 Atmospheric Density

<table>
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<tr>
<th>n</th>
<th>Altitude (m)</th>
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Table 4.1: Atmospheric Density data from “Space Mission Analysis and Design” [50].

As stated before in Section 2.2, atmospheric density is dependent on many constantly changing factors, such as the temperature of the atmosphere and the location of the sun in the sky. For this reason, it is impossible to hope to accurately predict the atmospheric density at any point around the Earth without the use of prohibitively complex atmospheric modeling. However, there exists accepted data on the general average densities one can expect to observe at certain altitudes above the Earth, as well as the statistical maximum and minimum densities. The handbook “Space Mission Analysis and Design” (SMAD) contains data on minimum, average, and maximum atmospheric densities for different altitudes, tabulated here in Table 4.1 on page 45 [50].

We can incorporate this data into our model by having the drag equations rely on the
4. DERIVING THE SOLUTION

minimum, average, and maximum density data independently; when we perform simulations, we shall run three separate orbit propagations. The minimum density data will be used for the first simulation, the average data will be used for the second, and the third will use only the maximum densities. In this way, we can determine the bounds of possible trajectories the satellite could statistically take.

The atmospheric density data from SMAD only has data for altitudes at increments of 50 km, as seen in Table 4.1. We can create an algorithm that interpolates between data points to find the atmospheric density at altitudes that are not given in the table. For example, if we want to find the minimum atmospheric density at 325 km, or even 325.856 km, we can use exponential interpolation to create an exponential curve that interpolates between the data points at altitudes of 300 km and 350 km. An exponential curve is the best for interpolation, because the density values decrease exponentially (as seen in Figure 2.10 on page 19).

Here, we will use an equation to find the density $\rho$ at a certain altitude $alt$ using exponential interpolation:\[\rho_{alt} = \rho_n \left( \frac{\rho_{n+1}}{\rho_n} \right)^{\frac{alt - n+1}{\Delta alt}},\]

where $alt$ is the altitude, $\Delta alt$ is the difference in altitude between successive rows in Table 4.1 (equal to 50,000 meters), $n$ is the row number (equal to $\left\lfloor \frac{alt}{\Delta alt} \right\rfloor + 1$), and $\rho_n$ is the $n^{th}$ density.

Let’s try this method of exponential interpolation, using 325 km as the altitude. With this, $alt = 325000$, so we can find the value for $n$:

$$n = \left\lfloor \frac{alt}{\Delta alt} \right\rfloor + 1 = \left\lfloor \frac{325000}{50000} \right\rfloor + 1 = 6.5 + 1 = 7.$$

If $n = 7$, we can choose our values of $\rho_n$ and $\rho_{n+1}$ from rows 7 and 8 from Table 4.1. If we are trying to find the minimum density $\rho$ at an altitude of 325 km, $\rho_n = 8.19 \cdot 10^{-12}$ kg m$^{-3}$ and $\rho_{n+1} = 2.34 \cdot 10^{-12}$ kg m$^{-3}$. Substituting this into our expression for density $\rho$ gives us

$$\rho_{alt} = \left( 2.34 \cdot 10^{-12} \right) \left( \frac{8.19 \cdot 10^{-12}}{2.34 \cdot 10^{-12}} \right)^{\left( \frac{325}{50} - 7 + 1 \right)} = 4.38 \cdot 10^{-12} \text{ kg m}^{-3}.$$
If we applied this interpolation algorithm to all altitudes from 0 to 1,000,000 meters, we would obtain curves that look exactly like those in Figure 2.10. Notice the exponential curves between each data point (these curves appear linear due to the logarithmic scale in the “density” axis).

4.3.5 The Orthogonal Projectional Area

As mentioned in Section 2.2 on page 19, a satellite can tumble in space, changing its orientation with respect to its direction of velocity. The satellite’s orientation will determine the orthogonal projectional area $A$ or reference area, the 2-D outline or “shadow” of the satellite that interacts with high-velocity particles.

When describing how an object can rotate in space, it is necessary to use consistent conventions; there are many ways to describe an object’s attitude, or orientation. If an object begins at a starting orientation, one could first rotate it around an axis in $x$-$y$-$z$ coordinates by a certain angle. This angle could be called $\alpha$; let’s say that we rotate the object about the vertical $z$-axis by an angle $\alpha$. Next, the object could rotate around a different axis (let’s say the $y$-axis) by some angle $\beta$. This might bring the object into some orientation $P$.

Now, let us switch the order in which we make these rotations: we shall rotate our object around the $y$-axis by angle $\beta$, then around the $z$-axis by angle $\alpha$. This creates a resultant orientation $Q$. The important thing to note is that the orientation $P$ is not the same as $Q$. These rotations, like functions, must be composed in the correct order to result in a particular rotation.

There are many conventions for describing rotations and orientations, organized into a function composition system called Euler angles. Since objects exist in three dimensions, there are usually three Euler angles of rotation required to fully describe orientation. The convention where we rotate in the $z$-axis and then $y$-axis would be the $ZY$ convention; to fully describe an orientation, we would have define another axis to rotate around. It could be the $x$-axis, where the convention would be $ZYX$. It could rotate around the $z$-axis again, so the convention would be called $ZYZ$. Or, we could define a new axis that stayed fixed to the orientation of the object, such as the object’s vertical axis, $Z'$. A rotation in $Z'$ would cause the object to spin in place like a top, regardless of its current orientation. Here, the convention would be called $ZYZ'$.

When describing the rotation of a satellite to find its current orthogonal projectional area, we can use Euler angles. However, we can make the simplifying assumption that the
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direction of the satellite’s velocity is parallel to a single axis. Here, we can say that \( \dot{\mathbf{v}} \) is parallel to the \( x \)-axis. This can allow us to fully describe the relevant orientation of the satellite with only two rotations, in the \( z \) and \( y \) axes. Using the \( ZY \) convention mentioned earlier, a rotation about the \( x \)-axis would not change the orthogonal projectional area, instead causing the area’s shape to rotate around in the \( y-z \) plane. The actual area of the orthogonal projection will remain constant for all rotations about the \( x \)-axis.

Now, we can begin to derive an expression for the orthogonal projectional area of the FIREBIRD satellite, given its orientation through the convention described. With a robust expression, we can always know what value to use for \( A \) given its current attitude.

4.3.5.1 GEOMETRY OF A CUBOID

We begin our derivation of the orthogonal projectional area of a CubeSat by first assuming that our CubeSat exists as a perfect cuboid. For example, a CubeSat of the FIREBIRD pair could be modeled as a cuboid with a square base \( w = 10 \text{ cm} \) and a height \( l = 15 \text{ cm} \). Ignoring the area of the exposed antenna and electronics, which is small compared to the rest of the satellite, will help simplify our geometric analysis.

We now can visualize a cuboid in space, rotated at an angle \( \alpha \) about the \( z \)-axis and then at an angle \( \beta \) about the \( y \)-axis. By assuming that the cuboid has a forward velocity \( v \) in the positive \( x \)-axis, we know that any atmospheric particles that strike the exposed area of the cuboid will be moving in the direction of the \(-x\)-axis from the cuboid’s perspective. Therefore, when viewing the shape of the cuboid’s orthogonal projectional area, we shall also look in the direction of the \(-x\)-axis. In this way, we know that the shape of the orthogonal projectional area is in the \( y-z \) plane, since it is orthogonal to the directions of velocity and viewing. Restating and summarizing our assumptions: If a rectangular cuboid has a square base with width \( w \) and a height \( l \), and is first rotated about the \( z \)-axis through an angle \( \alpha \) and then about the \( y \)-axis through angle \( \beta \), it will have some orthogonal projectional area \( A \) parallel to the \( y-z \) plane. We can see the shape that constitutes this area while viewing the cuboid from the \(-x\)-axis, as shown in Figure 4.2. Note that the original orientation of the cuboid with no rotation has the \( w \times w \) sides facing the vertical direction, with a single \( w \times l \) side facing the viewer.

Next, we look through the image of the cuboid in Figure 4.2 to see a transparent view showing the hidden corner, as seen in the left side of Figure 4.3. With this, we can start splitting up the image into manageable squares and right triangles. By drawing horizontal (\( \hat{y} \) direction) and vertical (\( \hat{z} \) direction) lines from all points that represent corners of the
4.3 Equations of the Model

Figure 4.2: View of the rotated cuboid from the $-x$-direction. The left image shows a non-rotated cuboid, and the right image shows a cuboid rotated through angles $\alpha$ and $\beta$.

real cuboid (shown in the right side of Figure 4.3) and by erasing any extraneous lines, we arrive at the shapes illustrated in Figure 4.4.

Figure 4.3: Showing transparent lines (left) and drawing horizontal and vertical lines from corners within the two-dimensional shapes (right).

These shapes correspond to recognizable sections of the cuboid’s geometry. For instance, if we name certain points by the letters $a$ through $h$, as defined in Figure 4.5, we find that $\overline{dg}$ is the horizontal component of the width of the closest side (mostly facing the viewer), while $\overline{dh}$ is the horizontal component of the side directly to the left of the closest side. In
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Figure 4.4: Modifying Figure 4.3 by drawing and erasing lines to split the shapes into squares and right triangles.

addition to these, there are many other relationships we can use to find the areas of these individual triangles and squares.

These lengths exhibit very predictable behaviors when the cuboid is rotated. The lines \( \overline{dg} \) and \( \overline{eb} \) are equal for all rotations, as are \( \overline{dh} \) and \( \overline{fb} \). We know this must be true due to the symmetry of the original cuboid; the opposite sides of the cuboid are parallel, and therefore turned the same angle away from the viewer. This makes the horizontal components of the opposite sides equal. Also, since these lengths measure horizontal components of the lengths only, they are completely dependent on rotations \( \alpha \) and invariant in rotations \( \beta \).

Let us now find the relationships between these lengths and angle \( \alpha \). We know that if the original cuboid’s sides each have a width \( w \), then the length of the horizontal component of the closest side, \( dg \), will be equal to \( w \) when \( \alpha = 0 \) radians, since the corresponding side is directly facing the viewer. Likewise, at this same angle, we know that \( fb = 0 \), since the corresponding side is facing completely perpendicular to the viewing direction. This is illustrated in Figure 4.6.

Now, by rotating the cuboid so that \( \alpha = \pi \) radians (or 90\(^\circ\)), we can see how \( dg \) and \( fb \) must change. The side corresponding to component \( dg \) is now completely turned away from the viewer instead of fully facing, while the side corresponding to component \( fb \) is now entirely facing the viewer. Therefore, we know that at an angle \( \alpha = \pi \), \( dg = 0 \) and \( fb = w \). Once again, this concept is illustrated in Figure 4.6.

Using this knowledge, we can create relationships between the lengths of lines \( \overline{dg}, \overline{fb}, \)
4.3 Equations of the Model

Figure 4.5: Showing points a through h on the shapes.

\[ \alpha = 0 \]

\[ \alpha = \frac{\pi}{2} \]

\[ \alpha = \pi \]

Figure 4.6: Showing cuboid where \( \beta = \pi \) and \( \alpha \) has various values ranging from 0 to \( \pi \) radians.
and \( \alpha \) using trigonometric functions that satisfy these conditions. We do not want negative lengths of these lines, so we make sure that the lengths are proportional to the absolute value of the trigonometric functions:

\[
dg = eb = w |\cos \alpha| \quad \text{and} \\
fb = dh = w |\sin \alpha|.
\]

For a quick reality-check, we can see that if we rotate the cuboid by an angle \( \alpha = \pi/2 \) radians, the two sides facing the viewer will have equal visible projectional areas, and therefore equal visible component lengths (shown in Figure 4.6). We can confirm this with our equations:

\[
w |\cos (\pi/2)| = w |\sin (\pi/2)| = \frac{w}{\sqrt{2}}.
\]

With a similar method for finding \( dg \) and \( fb \), we can find an expression for the lengths of lines \( \overline{ah} \), \( \overline{ae} \), \( \overline{fc} \), and \( \overline{fg} \), shown in Figure 4.5 on page 51. To visualize what these lines physically represent for the cuboid, it is best that we consider the special case where \( \beta = \pi \) radians. Here, the viewer sees the cuboid from a top-view, shown in Figure 4.6. We can see that \( ah = dg \), by the symmetry of the base of the cuboid. While observing the vertical line \( ah \), we may realize that it corresponds to a line drawn from the corner of the cuboid to a point closer to the center of its top area. When one views this line from above, where \( \beta = \pi \), one views it head-on; as our angle \( \beta \) decreases from \( \pi \) to 0, we are looking at the line from an increasing angle \( \pi - \beta \), and the visible line shortens in length. Therefore, when \( \beta = 0, ah = 0 \), and when \( \beta = \pi, ah = w |\cos \alpha| \).

So, we can now create an expression for the length of line \( \overline{ah} \), and by the same logical method, \( \overline{ae}, \overline{fc}, \) and \( \overline{fg} \):

\[
ah = fc = w |\cos \alpha \sin \beta| \quad \text{and} \\
ae = fg = w |\sin \alpha \sin \beta|
\]

Now, let us find the areas of the triangles \( \triangle dha, \triangle bfc, \triangle dgc, \) and \( \triangle bea \). Since \( \triangle dha \) and \( \triangle bfc \) each contain two corresponding sides of equal length, they are congruent; the same is true for \( \triangle dgc \) and \( \triangle bea \). So, the area of the triangles can be found, shown below.

\[
\triangle dha_A = \triangle bfc_A = \frac{1}{2} \cdot dh \cdot ah = \frac{1}{2} \cdot w |\sin \alpha| \cdot w |\cos \alpha \sin \beta| = \frac{w^2}{2} |\sin \alpha \cos \alpha \sin \beta|
\]

\[
\triangle dgc_A = \triangle bea_A = \frac{1}{2} \cdot eb \cdot ae = \frac{1}{2} \cdot w |\cos \alpha| \cdot w |\sin \alpha \sin \beta| = \frac{w^2}{2} |\sin \alpha \cos \alpha \sin \beta|
\]

Note the interesting equality; the four triangles all have the same area.
In Figure 4.5 on page 51, we can now look at the two unlabeled triangles directly below \( \triangle dgc \) and \( \triangle bfc \), respectively. Since these triangles have three sides that are equal in length to the sides of their corresponding triangle neighbor (\( \triangle dgc \) for the left triangle and \( \triangle bfc \) for the right), they must be respectively congruent to triangles \( \triangle dgc \) and \( \triangle bfc \). We now know the areas of all the triangles in the top half of the projected area shapes in Figure 4.5 by symmetry of the cuboid, the triangles and rectangle in the lower half must also be of the same area as those in the top. These equal areas can be visualized in Figure 4.7.

**Figure 4.7:** The distinct areas of the full projected area shape. Equal areas are organized by color.

To find the area of the blue rectangles (BR) in the top and bottom halves of the area in Figure 4.7, we can simply find expressions for the width and height of the rectangle in terms of our previously determined lengths, shown in Figure 4.5 on page 51:

\[
\text{BR}_A = ef \cdot eh \\
= (eb - fb) \cdot (ah - ae) \\
= (w |\cos \alpha| - w |\sin \alpha|) \cdot (w |\cos \alpha \sin \beta| - w |\sin \alpha \sin \beta|) \\
= w^2 |\sin \beta| (|\cos \alpha| - |\sin \alpha|)^2.
\]

Now only the areas of the green rectangles on the left and right sides and the center yellow rectangle remain to be found, shown in Figure 4.7. To find these, we must find a rule that will let us find the visible distance of line \( \overline{op} \). Since \( \overline{op} \) corresponds to the line along the height \( l \) of the cuboid, rotated about the \( y \)-axis by some angle \( \beta \), we can think of \( \overline{op} \) as the component of that rotated line in direction of the \( z \)-axis:

\[
op = l |\cos \beta|.
\]
Since we now have an expression for $op$, we can now find the area of the green rectangles (GR), shown in Figure 4.7. Notice that these rectangles have the height of a triangle directly above and below them, so that the total height of the rectangle-triangles system spans the height of $op$. The triangle above has a height of $ae$, and the triangle below has a height of $ah$. Therefore, an expression for the height of the green rectangle would be

$$GR_H = op - ae - ah$$

$$= l |\cos \beta| - w |\sin \alpha \sin \beta| - w |\cos \alpha \sin \beta|$$

$$= l |\cos \beta| - w |\sin \beta| (|\cos \alpha| + |\sin \alpha|).$$

Using this, we can find the areas of the green rectangles:

$$GR_A = GR_H \cdot dh$$

$$= (l |\cos \beta| - w |\sin \beta| (|\cos \alpha| + |\sin \alpha|)) \cdot w |\sin \alpha|.$$  

Notice that $GR_H$ is the height of the yellow rectangle (YR), shown in Figure 4.7.

$$YR_A = GR_H \cdot ef$$

$$= GR_H \cdot (eb - fb)$$

$$= (l |\cos \beta| - w |\sin \beta| (|\cos \alpha| + |\sin \alpha|)) \cdot (w |\cos \alpha| - w |\sin \alpha|).$$

Now, let’s add up all the areas together to find an expression for the total orthogonal projectional area $A$ of the cuboid:

$$A = 12 \cdot \triangle dgc + 2 \cdot BR_A + 2 \cdot GR_A + YR_A$$

$$= 12 \left[ \frac{w^2}{2} |\sin \alpha \cos \alpha \sin \beta| \right] + 2 \left[ w^2 |\sin \beta| (|\cos \alpha| - |\sin \alpha|)^2 \right]$$

$$+ 2 \left[ (l |\cos \beta| - w |\sin \beta| (|\cos \alpha| + |\sin \alpha|)) \cdot w |\sin \alpha| \right]$$

$$+ \left[ (l |\cos \beta| - w |\sin \beta| (|\cos \alpha| + |\sin \alpha|)) \cdot (w |\cos \alpha| - w |\sin \alpha|) \right]$$

$$= w^2 |\sin \beta| + w l |\cos \beta| (|\cos \alpha| + |\sin \alpha|).$$

We now have an expression for the area of the orthogonal projection (onto the y-z plane) of our rectangular cuboid with a base width $w$ and height $h$, given rotational angles $\alpha$ (about the z-axis) and $\beta$ (about the y-axis). This equation can be stated as a function $A(\alpha, \beta)$:

$$A(\alpha, \beta) = w^2 |\sin \beta| + w l |\cos \beta| (|\cos \alpha| + |\sin \alpha|). \quad (4.17)$$
4.3 Equations of the Model

4.3.5.2 The Average Area

While the orthogonal projectional area $A$ of our CubeSat can vary throughout its orbit for a variety of reasons, we don’t know the instantaneous rotation angles the satellite will have during its orbit. To compensate for this, we will use our derived expression for $A$ to find the average $A$ the satellite will have over all possible rotations. Finding this average value requires a bit of calculus. The general equation for the average value of a two-variable function of $x$ and $y$ over some intervals is:

$$f_{\text{avg}}(x, y) = \left( \frac{1}{d-c} \right) \left( \frac{1}{b-a} \right) \left( \int_c^d \int_a^b f(p, q) \, dp \, dq \right). \quad (4.18)$$

In this case, the intervals $[a, b]$ and $[c, d]$ constitute the possible angles of rotation for the cuboid in the $p$ and $q$ directions, respectively. We know that the cuboid can rotate freely through all angles; therefore, we shall set our intervals to span $[0, 2\pi]$ radians in both the $\alpha$ and $\beta$ directions.

We can now insert our function $A(\alpha, \beta)$ and simplify the definite integral:

$$A_{\text{avg}} = \left( \frac{1}{2\pi - 0} \right) \left( \frac{1}{2\pi - 0} \right) \left( \int_0^{2\pi} \int_0^{2\pi} A(\alpha, \beta) \, d\alpha \, d\beta \right)$$

$$= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} w^2 |\sin \beta| + w l |\cos \beta| (|\cos \alpha| + |\sin \alpha|) \, d\alpha \, d\beta$$

$$= \frac{w}{2\pi^2} \left[ w \pi \int_0^{2\pi} |\sin \beta| \, d\beta + 2l \int_0^{2\pi} (|\cos \alpha| + |\sin \alpha|) \, d\alpha \right]$$

$$= \frac{w}{2\pi^2} \left[ w \pi \int_0^{2\pi} |\sin \beta| \, d\beta + 2l \left( \int_0^{2\pi} |\cos \alpha| \, d\alpha + \int_0^{2\pi} |\sin \alpha| \, d\alpha \right) \right].$$

By symmetry, the remaining definite integrals can be evaluated by the use of the following trigonometric / integral equivalences:

$$\int_0^{2\pi} |\sin x| \, dx = \int_0^{\pi} \sin x \, dx + \int_0^{\pi} -\sin x \, dx = 2 + 2 = 4$$

$$\int_0^{2\pi} |\cos x| \, dx = \int_0^{\pi/2} \cos(x) \, dx + \int_{\pi/2}^{3\pi/2} -\cos(x) \, dx + \int_{3\pi/2}^{2\pi} \cos(x) \, dx = 1 + 2 + 1 = 4.$$

Substituting in the above equivalences can let us further simplify $A_{\text{avg}}$:

$$A_{\text{avg}} = \frac{w}{2\pi^2} \left[ \pi \, w \, 4 + 2l \left( 4 + 4 \right) \right]$$

$$= \frac{2w(\pi \, w + 4l)}{\pi^2}.$$

With this, we now have an expression for the average cross-sectional area of a rectangular cuboid, given the cuboid’s height $l$ and width of its square base $w$:

$$A_{\text{avg}}(w, h) = \frac{2w(\pi \, w + 4l)}{\pi^2}. \quad (4.19)$$
4. DERIVING THE SOLUTION

We can apply this equation to find the average cross-sectional area of the FIREBIRD satellites. Since this CubeSat type is $15 \times 10 \times 10$ cm, we can substitute in $w = 0.1$ m and $l = 0.15$ m:

$$A_{avg}(0.1, 0.15) = \frac{2 \cdot 0.1 \left( \pi \cdot 0.1 + 4 \cdot 0.15 \right)}{\pi^2}$$

$$= \frac{\pi + 6}{50 \pi^2}$$

$$\approx 0.018525 \text{ m}^2$$

$$\approx 185 \text{ cm}^2$$

4.3.6 ADDITIONAL SATELLITE SPECIFICATIONS

There are two more components of the drag equation that we must address: the drag coefficient, and the masses of the FIREBIRD satellites experiencing drag forces.

The drag coefficient $C_D$ is dependent upon a wide variety of factors, such as speed, shape, and surface area and texture of an object experiencing drag, as well as the atmospheric density of the air through which the object travels. It is usually an empirically determined value from experiments in wind-tunnels, but some drag coefficient estimates can be made from the flight data from a variety of satellite types. Based upon the performance of similar satellites, it is safe to assume the use of a rough coefficient of 4 will suffice for this simulation[50]. It is difficult to obtain a number with any additional accuracy, since the uncertainty of the coefficient increases with altitude[27].

While it is true that the drag coefficient is consistently uncertain for both identical satellites in a similar location, the separation speed can be strongly affected by the drag coefficient. Let us consider the hypothetical situation where the drag coefficients of both satellites are zero. Drag would not affect either satellite, and they would not separate at all. This allows them to stay within a relative distance of 100 km for an infinite amount of time. From this, it follows that an increase in their drag coefficients from zero, even if slight, will induce differential drag between the satellites, causing them to separate. A further increase in the drag will then cause them to separate faster. Therefore, for accurate modeling results, it is best to dial in the correct drag coefficients for both satellites, as accurately as possible. In lieu of this, choosing a drag coefficient of 4 gives us an acceptable upper-bound on the coefficients of the satellites, and any modeling results from this will correspond to the worst-case scenario of the satellites separating quickly.

For the mass of the FIREBIRD satellites, we shall initially use a value of 1.65 kg [26], as per the FIREBIRD structural specifications. When comparing orbital trajectories of
differently-massed satellites (and thus analyzing the satellite’s relative separation distance) we will induce differential drag into the satellite system by changing the mass of one satellite by a small increment. For instance, a possible mission scenario could require that the heavier satellite of the FIREBIRD pair weigh in at 1.65 kg, while the lighter satellite would have a mass of 1.645 kg.

Now that we have systems of equations describing the gravitational and drag forces that a satellite would experience around the Earth, it is time to solve them. It can be a long process to tie all the loose ends together. This is where we make the use of computer programming to perform many repetitive numerical calculations very quickly.

4.4 MATLAB Programming

Matrix Laboratory (MATLAB)\([12]\] is a program where one can utilize a high-level programming language to perform numerical computation. MATLAB is particularly adept at efficiently manipulating vectors and matrices\([\dag]\). Since the FIREBIRD satellite problem requires that we numerically solve complex differential equations and examine the resulting output, we must generate, store, and process long strings of data; each element in the strings will represent the satellite’s position and velocity at a certain point in time. This makes MATLAB an excellent programming language to use for this problem.

A special advantage to using MATLAB to do orbital simulations is that we can use its numerical differential equation solvers. These programs use very powerful numerical methods to solve differential equations. MATLAB programs like ODE45 and ODE113 use variable step-sizes and large-order methods to evaluate the derivatives in a differential equation with high accuracy\([40]\).

For my MATLAB approach to solving the FIREBIRD problem, I chose to implement the ODE113 solver; it can estimate the error of its evaluation at each step and choose an appropriate method, anywhere from first-order to thirteenth order, to use in the calculations. It can also vary the step-size of the calculations based on its estimation of the error. These two methods let the solver perform less-accurate calculations when the error is low to speed up the solving process, as well as increase the accuracy when the estimated error gets too large. This makes the solver one of the fastest and most robust methods that one can employ with the MATLAB language.

Portions of MATLAB code are included as an appendix.

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\(\dag\) Vectors are strings of ordered numbers, while matrices are organized square arrays of numbers.
CHAPTER 5

APPLICATIONS

5.1 TESTING THE MODEL

After creating our model with our derived equations of motion, the next step is to test its realism. We can do this by systematically testing the gravitational and drag portions of our model for different situations to see how the results compare to accepted preexisting models and measurements.

First, we can test the basic gravitational portion of our model, assuming a spherical Earth with no atmosphere. A good test is to see what the model predicts a satellite’s sidereal orbital period to be, given its orbit shape. This is the time it takes for a satellite to return to the same position in an orbit, relative to the stars. To simplify our analysis, we can assume that our orbit is circular. If this is the case, we can relate the velocity of the satellite with its acceleration with the equation for centripetal acceleration:

\[ a_c = \omega^2 r, \]

where \( \omega \) is the satellite’s angular velocity, and \( r \) is the distance from the satellite to the center of the Earth.

If we want to find the orbital period \( T \), we know that the satellite must circle the Earth once during the duration of that period. This makes \( \omega = \frac{2\pi}{T} \) radians per second:

\[ a_c = \left( \frac{2\pi}{T} \right)^2 r. \]

The centripetal acceleration \( a_c \) always points inward towards the Earth with a constant magnitude. Since the radius \( r \) remains constant for a circular orbit, we can use Newton’s Equation of Universal Gravity to find the inward-pointing acceleration due to gravity:

\[ \frac{\mu}{r^2} = \left( \frac{2\pi}{T} \right)^2 r. \]
Now, solving for the orbital period $T$ gives us

$$T = 2\pi \sqrt{\frac{r^3}{\mu}}.$$

(5.1)

Let’s choose a circular orbit within our altitude range of 0 to 1000 km; an altitude of 950 kilometers will suffice.

The radius of the Earth is 6378136.3 meters, so $r = 6378136.3 + 950000 = 7328136.3$ m. Also, the value for $\mu$ is defined as $3.986004415 \cdot 10^{14}$ m$^3$ kg$^{-1}$ s$^{-2}$ [5, 50]. Placing these values into Equation 5.1 gives us a predicted orbital period of about 6234 seconds, or about 1.7 hours.

To test the basic gravity equations of our model, we can run the numerical model in MATLAB with no spherical harmonic gravity or drag forces:

![Figure 5.1: Plot of a satellite's path in a circular orbit, computed during the testing of the model's basic gravity portion. The green/yellow dot corresponds to the starting/ending position of the satellite.](image)

At an altitude of 950 km, the results of the numerical computation predicted that the satellite would fly within 0.3 mm of its starting position at time $T$. The result for this test implies that the model for basic Newtonian gravity is working exactly as expected.

Now would be a good time to shift our testing over to the spherical harmonic gravity portion of the model. When testing harmonic gravity, we can observe the predicted pre-
cession rate of an orbit. For instance, an orbit at a certain inclination $i$ and elevation will precess in a certain way: the Longitude of the Ascending Node will rotate about the Earth with a constant angular velocity $\omega$.[23]

Derived from Newton’s equations of motion through perturbation theory[28], there is a closed-form expression for the precession rate of an inclined orbit, in units of radians per orbit:

$$\frac{\Delta \Omega}{\text{orbit}} = -\frac{3}{2} \frac{2 \pi J_2}{\mu r^2} \cos i,$$

where $\Delta \Omega$ is the change in the Longitude of the Ascending Node and $J_2$ is a measure of the Earth’s bulge, derived from the Earth’s flattening ($J_2 = 1.7555 \cdot 10^{10} \text{ km}^5 \text{ s}^{-2}$)[28].

If we assume that this orbit is circular, we can combine this with our expression for the orbital period in Equation 5.1 to obtain the angular velocity of the precession of the node:

$$\dot{\Omega} = \frac{\Delta \Omega}{\Delta t} = \frac{\Delta \Omega}{\text{orbit}} \frac{\Delta t}{\text{orbit}} = -\frac{3}{2} \frac{2 \pi J_2}{\mu r^2} \cos i \cdot \frac{1}{\left(2 \pi \sqrt{\frac{r^3}{\mu}}\right)} = -\frac{3}{2} \frac{J_2 \cos i}{\sqrt{r^7 \mu}}.$$ (5.2)

Now, let’s turn this angular velocity into the time $t_p$ it takes for an orbit to precess completely around the Earth:

$$\left( -\frac{3}{2} \frac{J_2 \cos i}{\sqrt{r^7 \mu}} \text{ radians/second} \right) \cdot (t_p \text{ seconds}) = 2\pi \text{ radians.}$$

Solving for a positive time $t_p$ gives us

$$t_p = \frac{4}{3} \frac{\pi}{J_2} \sqrt{\frac{r^7 \mu}{\cos^2 i}}.$$ (5.2)

Let’s test an orbit inclined at $20^\circ$ at a 400 km altitude, without drag forces to consider. According to our derived expression for $t_p$, it would take 48.6 days for the orbit to precess completely around the Earth.

We shall use the following spherical harmonics coefficients for our numerical model:

$$C_{2,0} = -1.0826269 \cdot 10^{-3} \text{ and } C_{3,0} = 2.5323 \cdot 10^{-6}$$ [2] [5].

Testing our numerical model for with these conditions yields the results shown in Figure 5.2. The orbit shown on the left figure shows a yellow dot marking the starting location of the satellite; here, the orbit is traveling to the right and moving from below the equator to above. After the twelve days of orbital paths shown, we can see the orbit precessing to the left. This is confirmed by Equation 5.2 when the inclination is between 0 and $90^\circ$, the direction of the precession is opposite the direction of the satellite’s motion.
5. APPLICATIONS

Figure 5.2: Path of a precessing orbit inclined at 20° at a 400 km altitude. The left figure shows the orbit after 12 days, and the right shows the orbit after 24 days. The yellow dot represents the starting location for the satellite. Note that the Earth is continually rotating throughout this orbit, and not held fixed as the figure might suggest.

After twelve more days, we can see on the right figure of Figure 5.2 that the point of the original starting location, which we shall call the ascending node, intersects with the current descending node. These nodes are on opposite sides of the Earth; as a satellite rises above the equator at a point, it will fall below the equator at a point 180° around the Earth. Since we now see the current descending node intersect the starting position, we know that the orbit has precessed half-way around the Earth in about 24 days. The full precession time was twice this, so this confirms that the orbit is precessing at the correct rate.

Using the same equation for $t_p$, we can try an additional circular orbit called a sun-synchronous orbit. Inclined at around 98° at an altitude of 700 to 1000 km, this useful orbit precesses around the Earth about once every year\[50\text{].} We can explore the calculated and simulated precession times for an orbit with an altitude of 800 km and inclination of 98.6°; this yields a calculated time of 365.4 days. With the same method of determining the simulated $t_p$, we find that the simulated and calculated times once again agree.

Finally, we can test the drag portion of our model. An easy way to confirm its accuracy is to determine how long a simulated satellite can remain in a low orbit before crashing to

---

\[1\text{A sun-synchronous can literally orbit with the sun. The FIREBIRD satellite mission will maintain a sun-synchronous orbit, as this provides the necessary inclination to access the poles of the magnetosphere and sufficient sunlight for solar power.}^\text{26}\]
5.1 Testing the Model

<table>
<thead>
<tr>
<th>Altitude (km)</th>
<th>SMAD (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Min. $\rho$</td>
</tr>
<tr>
<td>300</td>
<td>49.9</td>
</tr>
<tr>
<td>350</td>
<td>195.6</td>
</tr>
<tr>
<td>400</td>
<td>552.2</td>
</tr>
<tr>
<td>450</td>
<td>872</td>
</tr>
<tr>
<td>500</td>
<td>1205</td>
</tr>
</tbody>
</table>

Table 5.1: Partial deorbit time data from SMAD, when $C_B = 50$ kg m$^2$ [50].

Earth from excessive drag, or “deorbit,” given its initial altitude. We can then compare these times with accepted values from SMAD [50].

The tables of orbital decay times in SMAD (located in the back cover of SMAD and partially reproduced above in Table 5.1) are organized by altitude and a number called the ballistic coefficient $C_B$ of a satellite. Much like the drag coefficient $C_D$ in the drag equation, the ballistic coefficient determines how atmosphere will affect a high-velocity object:

$$C_B = \frac{m_{sat}}{AC_D}.$$

If we wanted to find the ballistic coefficient of a satellite, we would simply substitute its mass $m_{sat}$, reference area $A$, and drag coefficient $C_D$ into the above equation for $C_B$. For our analysis, we shall choose a ballistic coefficient $C_B = 50$ kg m$^2$ [50], as this value is used in the SMAD table and requires minimal computing time to verify with the numerical model.

Finding a drag coefficient of a satellite is not an easy task; for higher altitudes, the uncertainty of the coefficient increases greatly [27]. However, based on previous satellite data with similarly-shaped satellites, a drag coefficient of 4 will be sufficient for our simulations [1, 50], corresponding to the worst-case scenario. Using this value for $C_D$, as well as our determined average reference area $A = 0.018525$ m$^2$, we can find a satellite mass that will bring the ballistic coefficient to $50$ kg m$^2$; in this case, we will use a mass $m_{sat} = 3.7$ kg for our satellite.

Now, we can compare the data in SMAD, as shown in Table 5.1, with the results of our numerical modeling. Since calculating orbital trajectories for times past two years quickly becomes very computationally intensive for the numerical model, we shall forgo calculating the minimum deorbiting time at 450 and 500 km. Instead, we will test the remaining data by comparing minimum deorbit times at 300-400 km and maximum deorbit times at 300-500 km, with increments of 50 km for both minimum and maximum.
5. APPLICATIONS

Figure 5.3: Decaying (slightly eccentric) orbit of a satellite with an initial altitude of \(\sim 250\) km, subject to the maximum atmospheric density model and \(C_B = 50\) kg m\(^2\).

When a satellite decays, as shown in Figure 5.3, it eventually reaches a critical altitude where the satellite fully deorbits within hours. For this satellite, the simulated numerical model predicted that critical altitude to be about 150 km. The authenticity of this behavior is confirmed with data contained in SMAD; a satellite at an initial altitude of 150 km will deorbit within 4 hours[50].

Now, we can compare the SMAD deorbit times with the model deorbit times for various altitudes, compiled below in Table 5.2.

<table>
<thead>
<tr>
<th>Altitude (km)</th>
<th>SMAD (days)</th>
<th>Simulated (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Min. (\rho)</td>
<td>Max. (\rho)</td>
</tr>
<tr>
<td>300</td>
<td>49.9</td>
<td>11</td>
</tr>
<tr>
<td>350</td>
<td>195.6</td>
<td>30.9</td>
</tr>
<tr>
<td>400</td>
<td>552.2</td>
<td>77.4</td>
</tr>
<tr>
<td>450</td>
<td>-</td>
<td>181</td>
</tr>
<tr>
<td>500</td>
<td>-</td>
<td>393</td>
</tr>
</tbody>
</table>

Table 5.2: Deorbit time data from SMAD compared to results from numerical model.
The results from the numerical model are in general agreement with the accepted predicted values shown in SMAD. The predicted maximum deorbit times are consistent in magnitude, with errors ranging from ten to thirty percent, and the minimum times match very closely, with errors below ten percent.

After observing positive results from these tests, coupled with our careful approach in the derivation of the model equations, we can be reasonably sure of the validity of our model’s behavior. It is here where we can confidently explore the results of our model for the two separation strategies of the FIREBIRD mission.

5.2 EXPLORING THE RESULTS

The separation strategies outlined in Section 3.3 attempt to make use of differential drag to control the separation of the FIREBIRD satellites. The first strategy uses only differential drag to induce a separation, and the second both a spring and differential drag. We will attempt to simulate the physics of these scenarios with our MATLAB model to find the optimum and most realistic separation strategy for the FIREBIRD mission.

5.2.1 SEPARATION VIA DIFFERENTIAL DRAG

For this scenario, the FIREBIRD satellites forgo the use of any conventional separation methods involving springs. Instead, the individual satellites are separated through relative differences in acceleration due to drag acting on differently-massed satellites. As they exit the P-POD and initially fly together, the satellites are traveling at the same speed. However, the force of drag affects their motion, and they begin to feel a slightly different acceleration due to their differences in mass. The heavier satellite of the FIREBIRD pair has greater inertia and experiences smaller drag acceleration than the lighter satellite. This causes the satellites to separate throughout their orbit; the mission will end when they separate beyond a maximum relative distance (MRD) of 100 km.

For a time, the lighter satellite will fly directly behind the heavier satellite after their exit from the P-POD. During this stage, the lighter satellite could be said to be in the heavier satellite’s “shadow” of drag, where atmospheric particles that would normally apply drag forces to the lighter satellite have already been deflected by the heavier one. This will likely change the initial drag behavior in a way that cannot be fully predicted, but it can be surmised that it will somewhat reduce the differential drag between the satellites, causing them to fly more like a single entity rather than two separate satellites. However, we may
5. APPLICATIONS

Figure 5.4: Separation distance of the FIREBIRD satellites over time; note that this manifests itself as a range of possible curves, bounded by the solutions in minimum and maximum atmospheric densities.

proceed with the assumption that there is enough drag applied on the sides of the satellite (catching on the edges of solar cells and antennas) to provide an initial separation.

To test this separation method’s effectiveness, we can simulate the separation for a variety of orbital conditions. The FIREBIRD satellites will be in an approximate sun-synchronous orbit[26], so the method could be tested for various circular orbits with altitudes between 600 and 800 km and inclinations around 98°. We can also test various mass differences that will induce an acceptable separation velocity. For our simulations, we will have our heavier satellite’s mass remain constant at 1.65 kg, while the other satellite shall be anywhere from 5 to 10 grams lighter.

We created our MATLAB model to simulate one orbital path of a satellite, given its initial trajectory and the atmospheric drag its experiences. If we want to simulate the two FIREBIRD satellites separating from each other through differential drag, we can simply simulate the paths of the satellites one at a time. Both simulations can start in the same
position and initially travel in the same direction, but one of these will determine the orbital path of a lighter satellite. We must assume that the satellites cannot interfere with each other, so the actions of one satellite do not determine the trajectory of the other.

Since our atmospheric density model contains data for minimum and maximum atmospheric densities, as well as an average density, we can perform two additional simulations for each satellite in FIREBIRD. One simulation will use the minimum model of atmospheric density, and the opposite simulation will show how the separation strategy performs under maximum density. An average density will also be helpful in gauging the behaviors that fall in the middle between the two extremes.

As shown in Figure 5.4, we can compare the trajectories of each satellite as they move away from each other and find the distance between them at each point in time. Since the minimum and maximum atmospheric densities bound the behavior of the satellites, the possible separation rates and curves could be any one curve within the shaded area in Figure 5.4. It is difficult to predict exactly how the satellite will behave in an atmosphere; this introduces uncertainty as the length of the mission increases, leading to a range of possible curves.

We can proceed with our modeling by testing orbits with altitudes of 600, 650, 700, 750, and 800 km. For each of these, the mass differences between the FIREBIRD satellites can span five equally-spaced values between 5 and 10 grams: 5, 6.25, 7.5, 8.75, and 10 g. By simulating the satellites’ trajectories, we can observe when their separation distance exceeds the MRD of 100 km. Once the satellites’ MRD increases beyond 100 km, the data they gather will not be as useful and the mission will end.

The results of the simulations are compiled in Tables 5.3, 5.4, and 5.5 on page 68, showing the mission duration for a maximum, average, and minimum atmospheric density, respectively. Upon observation of the data, interesting trends present themselves; these trends can be confirmed as logical physical phenomena:

- The predicted mission duration for all altitudes and masses is longer for the minimum atmospheric density $\rho$ and shorter for the maximum $\rho$, while the durations for the average $\rho$ fall in between. This follows, since when the atmospheric density is assumed to be thinner than usual, the force of drag on both satellites will be less. As a result, the satellites will experience a smaller acceleration difference due to their differences in mass and will separate more slowly, increasing the time they are within the 100 km range. By the same logic, a thicker atmosphere leads to stronger drag forces on both
satellites, increasing the difference in acceleration they experience and causing them to separate faster.

- As the mass differences increase, the mission duration decreases. This is to be expected: since differently-massed satellites experience a corresponding difference in acceleration between them, a larger difference in mass will lead to an even greater difference in acceleration, causing the FIREBIRD satellites to separate faster and shortening the time their MRD is within 100 km.

- For greater altitudes, the mission duration is also greater. Once again, this makes sense: the atmospheric density $\rho$ is greatly reduced for increases in altitude, leading to weaker drag forces and therefore smaller differences in acceleration.

<table>
<thead>
<tr>
<th>Mass Diff. (g)</th>
<th>Altitude (km)</th>
<th>600</th>
<th>650</th>
<th>700</th>
<th>750</th>
<th>800</th>
</tr>
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<tbody>
<tr>
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<td></td>
<td>70</td>
<td>100</td>
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</tr>
<tr>
<td>6.25</td>
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<td>110</td>
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<td>70</td>
<td>100</td>
<td>130</td>
<td>170</td>
</tr>
</tbody>
</table>

Table 5.3: Simulated mission durations with Maximum $\rho$ (days)

<table>
<thead>
<tr>
<th>Mass Diff. (g)</th>
<th>Altitude (km)</th>
<th>600</th>
<th>650</th>
<th>700</th>
<th>750</th>
<th>800</th>
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<td>210</td>
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<td>370</td>
</tr>
</tbody>
</table>

Table 5.4: Simulated mission durations with Average $\rho$ (days)

<table>
<thead>
<tr>
<th>Mass Diff. (g)</th>
<th>Altitude (km)</th>
<th>600</th>
<th>650</th>
<th>700</th>
<th>750</th>
<th>800</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
<td>400</td>
<td>530</td>
<td>660</td>
<td>800</td>
<td>900</td>
</tr>
<tr>
<td>6.25</td>
<td></td>
<td>355</td>
<td>480</td>
<td>590</td>
<td>710</td>
<td>820</td>
</tr>
<tr>
<td>7.5</td>
<td></td>
<td>330</td>
<td>430</td>
<td>510</td>
<td>650</td>
<td>730</td>
</tr>
<tr>
<td>8.75</td>
<td></td>
<td>300</td>
<td>405</td>
<td>510</td>
<td>605</td>
<td>690</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>280</td>
<td>380</td>
<td>470</td>
<td>570</td>
<td>650</td>
</tr>
</tbody>
</table>

Table 5.5: Simulated mission durations with Minimum $\rho$ (days)
5.2 Exploring the Results

Figure 5.5: Simulated mission durations for various altitudes and mass differences.

From the data, we can see that it is desirable to choose orbits with high altitudes and to have minimum mass differences in order to maximize mission durations. There are, of course, limits on the altitude and mass difference that must be imposed. It may not be scientifically beneficial to have an altitude of 800 km or greater, and the act of removing a small amount of mass from a FIREBIRD satellite would be more difficult when smaller mass differences are desired\[26\]. Additionally, having a mission duration that is too long is not desirable; researchers may not want to wait a very long period of time to obtain data at all ranges of the 0-100 km separation distance.

Rather than attempting to maximize our mission duration, let’s pick out combinations of altitude and mass differences that predict reasonable mission durations. At an altitude of 700 km (the middle of our altitude test range), a mass difference of 5 g will separate the FIREBIRD satellites to a maximum allowed relative distance of 100 km in about 310 days under average atmospheric conditions, or in a little under a year. This mission duration has
Figure 5.6: Satellite separation over time for a sample scenario utilizing the spring/drag strategy. The solution curves at minimum (green), average (blue), and maximum (red) atmospheric densities highlighted.

an uncertainty of approximately eight months. Therefore, the lower bound of the mission duration is already predicted to be above four months, which was the proposed FIREBIRD mission length. If there was any issue with removing precisely 5 g from the designated lighter satellite (for example, if too much was removed), it has a small effect on the mission duration; having a mass difference of 6.25 g returns results that are similar enough to be acceptable mission durations.

The contents of Tables 5.3, 5.4, and 5.5 are visualized in Figure 5.5 showing additional interpolated mission durations.

5.2.2 Combination of Spring and Differential Drag

After the illuminating modeling results for the first separation strategy, it is natural to expect similar success for the second strategy, outlined in Section 3.3. However, since the strategy we are attempting to model uses both a spring and differential drag as separation mechanisms, our modeling efforts will yield little usable results. Because of the widely-varying atmospheric densities at altitudes around the Earth, the uncertainty in our solutions
would be too large to reasonably predict the FIREBIRD satellites’ behavior.

As a limitation of the capabilities of our model, we are not able to simulate satellite trajectories over all possible atmospheric densities, from minimum to maximum. We can only predict the trajectories for the special cases where the densities are always at the minimum or always at the maximum, and assume that the corresponding curves represent the boundaries where possible separation versus time curves exist. A breaking of this assumption can easily occur when the solution curves no longer act as strict boundaries to possible behavior, such as when solution curves for minimum, average, and maximum atmospheric densities intersect each other, demonstrated in Figure 5.6.

Let us look at the curve corresponding to the minimum atmospheric density, shown in green in Figure 5.6. There is an initial separation speed between the two FIREBIRD satellites from a spring pushing them apart, represented by the slope of the curve when the mission time is zero days. This initial speed remains more or less constant throughout its mission duration, which terminates after about 80 days. Clearly, the force of drag is not enough to overcome the initial separation velocity from the spring.

The curve corresponding to the maximum density (red) is very different than the previous curve. They start with the same separation velocity, but the force of drag is more than enough to overcome it; the leading satellite slows down quickly enough that the satellite’s relative speed is zero after about 30 days. The satellites then pass each other after 60 days and are separated even quicker by the continued drag forces; this causes them to pass the 100 km MRD at around the same time they would have with the minimum atmospheric density.

Finally, the behavior of the satellites for an average atmospheric density (blue) fully breaks our assumption that paths between minimum and maximum densities represent the boundaries of possible curves. The separation starts out with the same initial separation velocity, just as before; in this case, the force of drag is just enough to bring the satellites’ relative distance to the 100 km mark before they begin to double back after 150 days. Then, they pass each other at 320 days before separating again at 390 days. Clearly, the satellite behaviors at minimum and maximum densities do not bound the satellite behavior for all possible atmospheric conditions. This creates a large uncertainty in the separation distance over time, which means that our estimation for possible mission durations becomes less reliable.

Even if there was a way to reduce the uncertainty in our modeling, whether by somehow simulating more possible curves or by pinpointing some scenario that inherently reduces uncertainty (both difficult problems), we still cannot account for the uncertainty in the
separating behavior of the spring mechanism. The force that a spring may apply on the FIREBIRD satellites cannot be easily calibrated [17, 26]. As a result, even if atmospheric conditions are nominal for a particular initial separation velocity, they may not work for another unpredicted velocity from a misfiring spring. The satellites could separate differently than expected; regardless of whether they separate too quickly or slowly, any unexpected event may detrimentally affect the mission duration.

5.2.3 Implications of Results

For our efforts in modeling the first separation strategy, using only differential drag as a separation mechanism, we were rewarded with possible mission durations that corresponded and improved upon the proposed mission duration of the FIREBIRD project (four months). These durations could be reached by placing the FIREBIRD satellites into orbital conditions that are readily achievable; for example, one could use sun synchronous orbits with altitudes at 700 to 800 km and setting satellite mass differences to 5-6 grams. There are many other combinations of altitude and mass differences that could yield acceptable predicted mission durations.

This method of separation is an effective alternative to traditional spring-separation strategies, because it removes a large source of uncertainty: the applied force of the spring itself. Instead, this strategy relies directly on the laws of drag physics, applying a near-constant acceleration difference between the two satellites, causing them to separate in a controlled and understandable way. The bounds of satellite behavior, subject only to the separating force of drag, are well-known; from this, we can propose a reasonably certain mission duration, given the orbital scenario.

In light of these modeling results, it is my hope that the FIREBIRD project consider revising their separation strategy to rely upon the perturbative effects of differential drag. Doing so would simplify the separation process and remove any uncertainty in the forces applied from a spring mechanism, the behavior of which may not be predicted with enough reliable accuracy.
CHAPTER 6

FURTHER STUDY

There are many possible revisions and alterations one could include in a future model that would improve upon what we created. These improvements could both increase the model’s accuracy and broaden the range of possible orbital scenarios that could be explored. We shall discuss some of these improvements and how they would contribute to a more robust model of satellite behavior.

6.1 IMPROVED GRAVITY

To increase our gravitational model’s realism, we could take into account the pertubative effects of the Sun. Our current model assumes that the satellite rotates around the Earth alone, and that the inertial coordinate system lies at the center of the Earth. While this is a decent approximation for many applications, it is not a perfect description because the Earth (and therefore, the satellite) are revolving around the Sun. This creates a slightly different picture of gravity. These effects, as well as the non-negligible influence of the moon’s gravitational force, can be accounted for in a multiple-body gravitational model. In this model, the Earth and moon would be set in their correct orbits with respect to the Sun, and the simulated satellite revolving around the Earth would experience the appropriate gravitational forces from all three bodies in the system.

In our gravitational model, we used the second and third zonal spherical harmonic terms to provide a more accurate way to describe the gravitational field around the Earth. This will make the orbital trajectories more accurate out to hundreds of days by modeling orbital precession and other perturbations. However, we could further improve our model by adding a fourth zonal term, as well as tesseral and sectoral harmonics of various orders,
to the gravitational model. These harmonics will account for additional bumps and ripples in the Earth's gravitational field, yielding a model with more “resolution.”

Another modeling approach would make use of pre-made gravitational model, such as NASA’s 2008 Earth Gravitational Model[29]. It provides an extremely accurate description of the Earth’s gravity, using over 2000 spherical harmonic terms. While extremely accurate and able to predict true orbital trajectories with a high resolution, these models are very computationally intensive. Sticking to a model that uses a few terms would likely suffice for general purposes.

6.2 Empirical Atmospheric Models

Even in its most basic form, atmospheric density is notoriously difficult to model accurately. It is dependent upon position and time dependent factors, such as the location over the Earth’s surface, the current sun activity in its solar cycle, and the temperature and humidity of the atmosphere. As such, any models that must rely on knowing the atmospheric density at positions around the Earth must put up with a large amount of uncertainty in their results. Any results produced from a model are only as precise as its individual components.

Since our atmospheric model consisted only of the minimum average and maximum density at altitudes spaced 50 km apart, the uncertainty in the predicted mission durations was large. However, there are other approaches to modeling the atmospheric density that could tighten our precision. A complex empirical atmospheric model from NASA, known as NRLMSISE-00, can produce more precise atmospheric densities at all locations around the Earth, taking into account changing atmospheric conditions[31]. Rather than assuming that atmospheric densities will always be at a their minimum, average, or maximum levels for the entire duration of the mission (which would never occur in real life), we can use NASA’s accepted atmospheric model to gain a more accurate and precise description of the conditions one would expect during the FIREBIRD mission.

There are two ways to successfully integrate the NRLMSISE-00 atmospheric model into a MATLAB program. In the first method, since atmospheric model is time-dependent, we could pick a simulation date that would correspond to the approximate point in the solar cycle when the FIREBIRD satellites would fly. Then, we could precalculate densities at various locations around the Earth and organize these values in a three-dimensional matrix of longitude and latitude and altitude, or a four-dimensional matrix that includes the model’s simulation time (to account for changing atmospheric conditions due to the
rotation of the Earth). One could then interpolate between these points in the matrix, just as we interpolated between densities at various altitudes in our current model.

For the second method, we would still choose an appropriate simulation date for the atmospheric model. Then, instead of precalculating densities and interpolating between these values, we can feed the output of the atmospheric model directly into our MATLAB model. This could provide a very accurate atmospheric model directly integrated into our model’s programming; this approach would also allow the atmospheric model to vary its long-term time-dependent components as the simulation time increases, instead of interpolating between densities at a few fixed points in time.

The downside to integrating the NRLMSISE-00 atmospheric model directly into MATLAB is considerable; the need for MATLAB to access an outside program for every step of the simulation would substantially increase the required computation time. The computational power required to perform such an orbital simulation in any reasonable amount of time would be enormous, and far beyond the capabilities of any personal computer. Therefore, it is more prudent to organize precalculated densities into an appropriate matrix; it is a sacrifice we must make in order to quickly obtain predicted results from our improved model.

6.3 Non-Empirical Drag Coefficient Determination

For our model, we didn’t go into much detail about finding a drag coefficient. To simplify our model, we settled to choose a drag coefficient of 4 in order to ensure that the simulation results corresponded to the worst-case scenario, where drag forces would strongly affect both satellites and cause a quicker separation for lower altitudes. In doing so, we add an unknown amount of uncertainty to our results for the mission duration. To reduce this uncertainty, we must find a more accurate number for the drag coefficient of the FIREBIRD satellites.

Traditionally, drag coefficients are educated guesses at an unknown value that describes how an object interacts with its surrounding atmosphere at high velocities. There are ways to empirically determine an approximate drag coefficient of an object in a wind tunnel, but wind tunnels cannot replicate the thin atmospheric conditions of an orbit, nor can they predict how the drag coefficient may change with altitude[27].

Nevertheless, there are some models in development which may have the ability to determine a particular object’s drag coefficient, simply due to the object’s shape and environment[27, 32]. By simulating a number of atmospheric particles striking an object at high velocity and studying the associated energy losses, the models can produce surprisingly
accurate predictions for the drag coefficient. Then, the uncertainty of the coefficient can be
made known through the normal program processes\[32\].

6.4 **Magnetic Modeling**

6.4.1 **Attitude Control and the Magnetic Field**

In order to transmit data back to a ground station, satellites must point their antennas in
the ground station’s general direction. In order to point their antennas, satellites must have
a particular attitude, or orientation. For the FIREBIRD mission, one of SSEL’s primary
objectives is to know the satellites’ approximate attitude throughout their orbits, or at
least to understand the basic rules that govern their attitude\[26\].

Like many small satellites, FIREBIRD uses passive attitude control so the satellite
does not tumble randomly throughout its orbit. This means that each satellite has one or
several bar magnets inside; these magnets apply torque to the satellites in the presence of
the Earth’s magnetic field, aligning the satellites in the field’s general direction. In practice,
the attitude of a satellite with passive attitude control will oscillate around the direction
of the Earth’s magnetic field at any point in its orbit, but never quite settling down so
that the attitude simply matches the current direction of the magnetic field. It is through
“attitude stabilization” that these oscillations are minimized, so that the attitude is more
or less continuously aligned with the magnetic field\[36\].

Much like the gravitational field of the Earth, the Earth’s magnetic field can be ac-
curately described through a series of spherical harmonics of the magnetic potential field.
The IGRF Geomagnetic Model is an excellent application of this, and outputs accurate
magnetic force vectors at any point around the Earth\[11\]. The direction of these vectors
would determine the torque on the bar magnets within the satellites. Incorporating this
into our current model and simulating would reveal the satellite’s resultant change in atti-
dude through time. Once again, we could use the magnetic model in the same ways as the
extensive NRLMSISE-00 atmospheric model, by either precalculating values and organizing
data into a multidimensional matrix for interpolation, or by feeding results directly from
the magnetic field model into the simulation.

Incorporating a model of the magnetic field and determining its relationship with the
satellite’s attitude is useful for the FIREBIRD problem: the satellite’s angles of orientation
with respect to their direction of motion through the atmosphere determines the satellite’s
instantaneous orthogonal projectional area \( A \). The satellites, rather than tumbling about
randomly through all possible angles of orientation, can now be described as having one particular attitude at any point in time. This gives them a unique value for $A$ to use in the drag equation at each point in time, rather than the average $A$ we previously used in our model.

Taking this idea further, it is possible to model how a change in the orientation of the bar magnets inside one of the FIREBIRD satellites would affect its resultant attitude with respect to the magnetic field, and therefore with respect to its direction of velocity through the atmosphere. This would change the behavior for its values of $A$ throughout its orbit, possibly causing the average $A$ of one satellite to be different than the other satellite. A difference in $A$ constitutes a difference in their accelerations due to drag; this would induce a separation without changing their masses. From this, one can realize the usefulness of the equation we derived for the orthogonal projectional area given its rotation angles; a test of a separation strategy like this could not be performed without the use of such an equation for $A$.

To fully perfect an attitude / magnetic field model, it would be prudent to model the stabilization of the attitude with respect to the direction of the magnetic field. With a passive attitude control system, stabilization of the attitude can be achieved with the use of hysteresis rods, which are magnetic rods that exhibit behaviors akin to having a magnetic “memory.” That is, these magnetic rods change their magnetic field in the presence of another magnetic field, such as the Earth’s. This acts as a sort of “drag” for a satellite’s rotation. Rather than continuously oscillating around the direction of the magnetic field, the attitude’s oscillations will lose energy due to the magnetic hysteresis rods; after a short period of time in orbit, any initial oscillatory behaviors are minimized and the attitude stabilizes[24, 36].

Including a model of attitude stabilization through magnetic hysteresis would show FIREBIRD engineers exactly how to incorporate bar magnets and hysteresis rods into the structure of the satellites. It would also predict a timeframe for how quickly the hysteresis effect would stabilize the attitude towards a particular direction. It is useful to know how long stabilization will take, because the satellites must be consistently pointing in certain directions in order to send and receive data via their antennae.

6.4.2 Magnetic Separation Mechanism

In Section 3.2 it was mentioned that a possible separation strategy could employ the use of small magnets in the feet of the FIREBIRD satellites as a separation mechanism. If
identically-poled magnets in the feet are placed next to each other within the P-POD, they should repel each other upon the satellite’s ejection. While ordinary metal springs could apply unpredictable forces for a variety of reasons\[1\], the forces applied by such magnets will always remain constant and very controllable. However, without proper analysis with a robust attitude / magnetic field model, it would be unclear as to how the inclusion of feet magnets would affect the satellite’s overall attitude behavior with respect to the magnetic field. Instead of using only bar magnets, we would have to simulate how feet magnets would augment passive attitude control, for better or for worse.

Since the force from a magnet extends over distance, the FIREBIRD satellites that separate from the feet magnets mechanism are still, for a time, in close enough proximity to each other to interact magnetically. While it is true that the magnets will apply a continuous repulsive force throughout the duration of the satellite’s initial separation, it is also possible that either the feet magnets or the bar magnets of each satellite could interact with the other in an attractive way. For example, one could visualize a situation where a misplaced magnet could inadvertently apply a torque upon one or both of the satellites, resulting in an attractive force between the satellites. If this happened, it is likely that the FIREBIRD satellites would either separate along an incorrect trajectory instead of along their direction of motion, or even worse, fail to separate and stick together for the entire duration of the mission\[17, 26\].

If considering a magnetic separation mechanism, it is important to completely understand all associated physical phenomena. A possible model could simulate the orbital trajectories of both FIREBIRD satellites as they leave the P-POD and separate via the feet magnets, until they separate to such a distance that the magnetic interaction between them becomes negligible. Then, the remainder of their orbital trajectories could be predicted individually, as with our current model. This would help ensure that an appropriate placement of feet magnets could be found, in a configuration that would not result in a mission-ending disaster.

---

\[1\]One reason for the unpredictable behavior of springs stems from how the relationship between the stress and strain of the metal may change over time during periods of prolonged compression within a P-POD\[17, 26\].
This appendix includes a small portion of the MATLAB code that was directly used in the ODE113 solver. It contains the basic gravitational and drag differential equations that were solved. This code calls several other functions not included here; it would be unwieldy to include every line of the multiple programs used for this thesis.

```plaintext
function a = fbdiff(t,y)
% ODE solver input function.
% Gravitational Constants (www.cdeagle.com/pdf/gravity.pdf; pg 9)
GME = 3.986004415e14; % Earth Gravitational Constant
C20 = -1.0826269e-3; % Zonal 2,0
C30 = 2.5323e-6; % Zonal 3,0
EqR = 6378136.3; % Earth Semi-major Axis, or Equatorial Radius

SD = 86164.09054; % Sidereal Day - Earth Rotation

csa = 0.0185247; % Cross-sectional Area in v_hat direction (m^2)
cd = 4; % Drag Coefficient

X = y(1); % x position / px
Y = y(2); % y position / py
Z = y(3); % z position / pz
vX = y(4); % x velocity / vx
vY = y(5); % y velocity / vx
vZ = y(6); % z velocity / vx

dnum = y(7); % Density (1=Min, 2=Average, 3=Max)
msat = y(8); % Mass of Satellite (kg)

% PRECALCULATION FOR PERFORMANCE

hran = 2*pi*(t)/SD; % Hour angle - Earth Rotation
ECEF = eci2ecef([X;Y;Z],hran); % Converting ECI to ECEF
Xp = ECEF(1); % ECEF X coordinate
```
Yp=ECEF(2); % ECEF Y coordinate
Zp=ECEF(3); % ECEF Z coordinate

[lat, lon, alt] = cart2geo(Xp,Yp,Zp); % Converting ECEF to Geodetic

den = atmden(alt,dnum); % Inserting Atmospheric Density

rad2 = X^2+Y^2+Z^2; % px^2+py^2+pz^2
rad = sqrt(rad2); % sqrt(px^2+py^2+pz^2)

vr=sqrt(vX^2+vY^2+vZ^2);

%[vTHETA,vPHI,vr] = cart2sph(vX,vY,vZ);

gr2=GME/rad2;
er=EqR/rad;
er2=er^2;

zr=Z/rad;
zr2=zr^2;
zr3=zr2*zr;
zr4=zr3*zr;

gec2=gr2*er2*C20;
gec3=gec2*er*(C30/C20);
gec2frac=gec2*(7.5*zr2-1.5);

gs=-(gr2+gec2frac+gec3*(17.5*zr3-7.5*zr));
ad=(csa*den*cd*vr)/(2*msat);

a = [...
   vX;...
vY;...
vZ;...
gs*X/rad-ad*vX;...
gs*Y/rad-ad*vY;...
-gr2*zr-(gec2frac-3*gec2)*zr-gec3*(17.5*zr4-15*zr2+1.5)-ad*vZ; ...
0;...
0 ...
];

dend
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