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The Role of Math Modeling in the Common Core

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The Role of Math Modeling in the Common Core

A Review of Math Modeling in the K-12 Classroom

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This thesis for honors recognition has been approved for the Department of Mathematics, Engineering and Computer Science

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ABSTRACT

Mathematics education in the United States has been through many reforms, the latest of which is the Common Core State Standards Initiative. While many of the old standards are modifications of the existing standards, the process of mathematical modeling introduced by the Common Core State Standards for Mathematics is one of the major changes. Modeling is a way to link the classroom to real world problems and enables students to problem solve, think critically, collaborate with other students, and use a variety of mathematical tools to answer problems. We first explore modeling within the Common Core State Standards Initiative for Mathematics since it is not only one of the eight standards for mathematical practice, and one of six domains in the high school, but also because modeling is integrated throughout the other domain’s standards. We then explore how modeling will be assessed, examine problems that could be used within the classroom, and discuss why modeling is an effective framework for a classroom’s curriculum.
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INTRODUCTION

Mathematics influences a wide variety of decisions made in society, from engineering to computer programming, from weather prediction to risk assessment, or even to optimizing semi-truck routes. Because of this variety of mathematical applications, mathematics is often likened to an art—by people who know mathematics—for its beauty and interconnectedness. Martin Gardner said, “All mathematicians share…a sense of amazement over the infinite depth and mysterious beauty and usefulness of mathematics” (1). In fact, many famous artists, including Leonardo da Vinci used mathematical computations in their paintings. Da Vinci used the Golden Ratio in the form of Golden Rectangles (rectangles whose length to width ratio is nearly equal to Phi). The Golden Ratio can be seen in his painting Mona Lisa. Furthermore, it can be found repeatedly in other types of paintings, in sculptures, and in architecture.

Teaching students to appreciate the artistic application/perspective of mathematics is an integral part of teaching mathematics. Students can be taught to appreciate the art of mathematics through modeling. Modeling is using mathematics to represent real world problems so that we can analyze or make predictions about the problem. Modeling helps them see the beauty of mathematics and appreciate its importance in their everyday lives. Furthermore, modeling with mathematics is an effective way to make students think critically and to problem solve. It also exposes students to the interconnectedness and beauty of mathematics.

The biggest difference between art and mathematics is accessibility. A beautiful painting can be appreciated by non-artists, but rarely can non-mathematicians appreciate the beauty, depth, and interconnectedness of mathematics, which requires a deep understanding of mathematics. A deep understanding does not come through rote memorization; it comes from modeling. According to Paul Lockhart in A Mathematician’s Lament, the American education system fails: [when educators] “concentrate on what, and leave out why, mathematics is reduced to an empty shell” (2).
He goes on to say “the art is not in the truth but in the explanation, the argument” (2). In other words, when students are given formulas and step-by-step directions to problems and the instruction ends with an exam, very little attention is paid to the intrinsic connections and beauty that mathematics provides. This type of instruction is in essence killing the students’ ability to problem solve, think critically, and persevere in solving the problems.

In recent years, leading educators have recognized the need to establish stricter standards and educational accountability. American students have notoriously scored lower in mathematics than their international counterparts. According to Craig Jerald (2008), in his report *Benchmarking for Success: Ensuring U.S. Students Receive a World-Class Education*, “American 15-year-olds ranked 25th in math and 21st in science achievement on the most recent international assessment conducted in 2006” (3). Furthermore, college professors and employers have seen a notable decline in many of the skills necessary to succeed in mathematics and in life, such as thinking critically and problem solving. In order to address these shortcomings, there have been many educational reform movements within the last decade both national and state-wide. One of the national reform movements, the No Child Left Behind Act (NCLBA) of 2001, holds schools accountable for annual yearly progress though national testing. The NCLBA allowed individual states to set their own standards. Although the NCLBA intended to increase accountability through national standardized testing, results show that teachers have often taught to the test rather than engaged students in challenging activities that might not be tested. In fact, these tests have revealed large disparities among states because of the different standards to which students are held. In a 2009 speech at the National Press Club, U.S. Secretary of Education Arne Duncan accused states of setting the bar too low in order to comply with the regulations of the NCLBA:

> We want to raise the bar dramatically in terms of higher standards. What we have had as a country, I'm convinced, is what we call a race to the bottom. We have 50 different standards, 50 different goal posts. And due to political pressure, those
have been dumbed down. We want to fundamentally reverse that. We want common, career-ready, internationally benchmarked standards. (4)

To meet Secretary Duncan's challenge, in 2009 the National Governors Association (NGA) and Council of Chief State School Officers (CCSSO) collaborated with teachers, researchers, and education experts to develop the Common Core State Standards Initiative, more commonly referred to as the Common Core State Standards (CCSS). This paper discusses the development of the CCSS for mathematics, which includes modeling with mathematics. It defines modeling, reviews the Smarter Balanced assessment, and reviews modeling problems that can be used within the classroom.

THE COMMON CORE

The CCSS are intended to provide educators with a consistent and clear idea of the content and skills K-12 students are expected to learn in English and mathematics. CCSS emphasizes critical thinking and learning the “why” not just the “what” in mathematics. In a statement on the CCSS standards, Secretary Duncan (2010) said,

The [common core] will help teachers, students and parents know what is needed for students to succeed in college and careers, and will enable states, school districts and teachers to more effectively collaborate to accelerate learning and close achievement gaps nationwide. (5)

The CCSS have been independently adopted by 44 states; as of March 2014, only Texas, Nebraska, Alaska, Virginia, Minnesota (Adopted English not Mathematics), Indiana, and Puerto Rico have not adopted the common core. In November 2011, Montana was one of the last states to adopt the standards for Mathematics and English. According to the CCSS, the standards aims to

…provide a consistent, clear understanding of what students are expected to learn, so teachers and parents know what they need to do to help them. The standards are designed to be robust and relevant to the real world, reflecting the knowledge
and skills that our young people need for success in college and careers. With American students fully prepared for the future, our communities will be best positioned to compete successfully in the global economy. (6)

In other words, the CCSS aims to provide the same common high standards throughout the United States so that a student in an inner city such as Los Angeles, California receives the same educational opportunities and is held to the same standards as a student in a rural area such as Melstone, Montana. Furthermore, the consistent standards ensure continuity when students move from state to state. In those instances, students will not be behind if one state teaches a concept such as fractions in the 5th grade and another state teaches it in the 6th grade.

**Development of the Common Core**
Before creating the Common Core in 2009, the NGA and CCSSO consulted with educational researchers to develop the standards. The NGA and CCSSO created the Common Core State Standards for Mathematics (CCSSM) based on surveys of required skills for entering college and workforce training programs, on assessment data, on comparisons to standards from high-performing states and nations, and on findings from the Trends in International Mathematics and Science Study (TIMSS) and other studies. These studies concluded that the traditional U.S. mathematics curriculum had to become substantially more coherent and focused in order to improve student achievement (6).

Prior to the time the NGA and CCSSO developed the CCSSM, they funded the research which can be found in the international benchmark report *Benchmarking for Success: Ensuring U.S. Students Receive a World-Class Education* (2000). This report recommended five actions to enable the United States to improve education and in turn become more competitive within the ever-changing global economy. The report urged math education reform since the authors found that “higher math performance at the end of high school translates into a 12 percent increase in future earnings” (3). This increase
in earnings is because higher performance better prepares students for college and workforce demands. The report also recommended these actions:

**Action 1:** Upgrade state standards by adopting a common core of internationally benchmarked standards in math and language arts for grades K-12 to ensure that students are equipped with the necessary knowledge and skills to be globally competitive.

**Action 2:** Leverage states’ collective influence to ensure that textbooks, digital media, curricula, and assessments are aligned to internationally benchmarked standards and draw on lessons from high-performing nations and states.

**Action 3:** Revise state policies for recruiting, preparing, developing, and supporting teachers and school leaders to reflect the human capital practices of top-performing nations and states around the world.

**Action 4:** Hold schools and systems accountable through monitoring, interventions, and support to ensure consistently high performance, drawing upon international best practices.

**Action 5:** Measure state-level education performance globally by examining student achievement and attainment in an international context to ensure that, over time, students are receiving the education they need to compete in the 21st century economy. (3)

Talks of the Common Core began in 2008; in 2009 the NGA and CCSSO met and began to develop the CCSS, which were finalized and released in June of 2010 for the individual states to adopt.

**How the CCSSM Differs from Previous Standards**
After the standards were announced, some of the opponents of the CCSSM argued that the standards are just repackaged from the current standards, or that they will bring all the state standards down to the lowest common denominator. In answer to this criticism, the NGA and CCSSO maintained that the new standards will require states to increase their standards:

The standards are designed to build upon the most advanced current thinking about preparing all students for success in college and their careers. This will result in moving even the best state standards to the next level. In fact, since this work began, there has been an explicit agreement that no state would lower its standards." (7)

Although some opponents to the CCSSM are right about the repackaging of standards—many areas that have been important in math will always be important—they miss the most important change that the CCSSM will bring: modeling with mathematics. In addition, critics assume that common standards must adhere to the lowest common denominator rather than considering that the new standards will require states to improve math education.

To guide states in improving math education, the NGA and CCSSO identified eight standards for mathematical practice that all mathematics educators should strive to develop in their students. According to the Common Core website, those standards are

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique students.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structures.
8. Look for and express regularity in repeat reasoning. (6)

(For a detailed description of the eight standards for mathematical practice, see Appendix B.)

Some of these standards for mathematical practice are already employed in many classrooms, such as number 1, make sense of problems and persevere in solving them and number 6, attend to precision. Other standards, such as number 5, use appropriate tools strategically and number 7, look for and make use of structures, may be easily implemented with little training. One of the greatest paradigm shifts is number 4, model with mathematics. Modeling with mathematics is something many math educators have had little to no exposure to or training with, and so they will need assistance to implement a curriculum that entails it. In order to teach modeling in the classroom, we must first understand what modeling is, consider how it will be assessed, determine when and how often modeling can be used, and decide what it should look like in our classrooms.

MODELING WITH MATHEMATICS

What is Mathematical Modeling?
Modeling with mathematics is taking a real world problem and finding optimal solution(s) using mathematics. For example, students might model this problem:

How many pizzas and of what type should you order for the conference?

To solve the problem, students must use specific mathematical skills. For example, they have to make assumptions such as assuming the number of people at the conference, the type of pizza most people would like, and how many pieces of pizza each person would eat on average. This type of modeling problem represents a simple type to solve. It can then be made more complicated by requiring students to make more advanced assumptions, such as incorporating the probability of a person eating a specific type of pizza, or the amount each person eats given they are male or female, or cost as a factor.
Regardless of the complexity, these types of modeling problems require students to take a real world problem, ask the question in mathematical terms, make simplifying assumptions, and solve the problem.

More formally, mathematical modeling is the process of using mathematical structures—equations, graphs, tree diagrams etc.—to represent real-world situations. A model reduces a real-world problem to its essential characteristics and allows students to infer, predict, and connect and test ideas. According to Mark Meerschaert (2010) in *Mathematical Modeling*,

> Mathematical modeling is the link between mathematics and the rest of the world. You ask a question. You think a bit, and then you refine the question, phrasing it in the precise mathematical terms. Once the question becomes a mathematics question, you use mathematics to find the answer. Then finally, you have to reverse the process, translating the mathematical solution back into a comprehensible, no nonsense answer to the original question. (8)

The National Council of Teachers of Mathematics (NCTM) defines and explains modeling as identifying and selecting relevant features of a real-world situation, representing those features symbolically, analyzing and reasoning about the model and the characteristics of the situation, and considering the accuracy and limitations of the model. NCTM suggests that middle school aged students use linear functions to model a range of phenomena and explore some nonlinear phenomena. High school students should study modeling in greater depth, generating or using data and exploring which kinds of functions best fit or model those data. (9)

As both Meerschaert's and NCTM's definitions indicate, teaching students to model with mathematics enables them to make links from the classroom to real world problems. It forces students to think critically, make sense of problems, and persevere in solving them. One could argue that if modeling is used effectively within the classroom, students can easily be shown to practice the other seven standards: make sense of problems and
persevere in solving them; reason abstractly and quantitatively; construct viable arguments and critique students; use appropriate tools strategically; attend to precision; look for and make use of structures; and look for and express regularity in repeat reasoning.

**Math Modeling in the Common Core**
Math modeling as an approach to meeting Common Core standards appears in two places in the CCSSM. While other domains, such as geometry, can be treated independently, the modeling domain must be applied within other domains. That is, modeling standards are integrated into the other high school math standards. (For a list of the standards that involve modeling, see Appendix A.)

While there are many different accepted modeling processes, NGA and CCSSO (2012) summarize the basic modeling cycle (i.e., steps the problem solver must follow) within the high school modeling standard:

1. identifying variables in the situation and selecting those that represent essential features,
2. formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables,
3. analyzing and performing operations on these relationships to draw conclusions,
4. interpreting the results of the mathematics in terms of the original situation,
5. validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
6. reporting on the conclusions and the reasoning behind them. (6)
A summary of the CCMS modeling cycle can be found in Figure 1 below.

![CCMS Modeling Cycle](image)

**Figure 1: CCSSM Modeling Cycle (6)**
According to this model, the problem step involves defining the problem and identifying variables. The formulating step involves formulating a model and making assumptions. After formulating a model, the process involves computing and answering, interpreting the results, insuring the results’ validity, reformulating model if the results are not valid, and reporting the results when they are.

In *Mathematical Modeling*, Meerschaert (2010) used a more concise five-step method to modeling that may be easier to understand and use:

1. Ask the question.
2. Select the modeling approach.
3. Formulate the model.
4. Solve the model.
5. Answer the question. (8)

He states that before we are even able to “ask the question” we must know the problem and then make some simplifying assumptions, list/define our variables, and write any equations/inequalities relating our variables and assumptions. We are then able to ask the question, which involves writing out our objective using mathematical language. Meerschaert stresses that after the model is solved it is very important to interpret the results and insure the conclusion in an easy-to-comprehend non-mathematical language.
Other modeling methods exist, most of which break the process into four to seven steps and name or emphasize slightly different steps. For example, a process that focuses more on reasoning has four components: Description, Manipulation, Translation or Prediction, and Verification (10). Mathematicians will have their own opinions of which process is best, but in this paper, we will use Meerschaert's five-step method when examining modeling problems.

In addition to establishing the steps students need to follow to model, the NGA and CCSSO also suggest that teachers require their high school students to model

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing stopping distance for a car.
- Analyzing savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Relating population statistics to individual predictions. (6)

This list is not all inclusive; any variety of real-world problems can be used in math modeling. While the CCSSM lists modeling in the two places (the eight standards for mathematical practice and in one of six of the high school domains) and suggests topics for modeling, they do not specify how much emphasis should be placed on modeling. This lack of specificity poses many questions not addressed by the CCSSM. How will it be assessed? How much modeling will actually be in the state’s assessment? How much
modeling should be integrated within the classroom? There is no consensus on any of these questions, so we will explore them in the next section.

**ASSESSMENT OF MODELING**

Although the importance and value of mathematics modeling is clear, appropriate methods of assessing modeling are somewhat unclear. However, since 2010, there have been several efforts to establish appropriate assessment methods for the CCSSM, including modeling. According to a press conference in 2010, Secretary Duncan awarded the Partnership for Assessment of Readiness for College and Careers (PARCC) and the Smarter Balanced Assessment Consortium (SBAC) a $330 million federal grant as a part of the Race to the Top Initiative (11) to develop a new generation of tests that are aligned with the CCSSM. Montana is among 30 other states that make up the Smarter Balanced Assessment Consortium. For this reason, this section focuses on the Smarter Balanced Assessment and their efforts to establish appropriate CCSSM assessment.

According to SBAC, their assessment uses computer adaptive technology. This means each question’s difficulty is based on the students’ previous answer(s) (12). The assessment includes two parts. Part one contains multiple choice/constructed response questions; part two requires students to complete a performance task. By design, students complete the assessment on two separate days. They complete part one during a specific block of time on day one. They complete part two, the performance task on a separate day, after teachers give instructions involving that performance task.

Sample assessment questions have been released, and the final assessment instrument is being prepared to be used in the 2014-2015 school year. As part of this preparation, several steps are being taken to ensure reliability:

- **Cognitive Labs:** Test developers receive feedback from individual students about their experience with the innovative test questions, accommodations for students with special needs, and the testing software.
• Small-scale Trials: Promising types of questions and software features are being piloted with hundreds of students.
• Pilot Tests: Students at approximately 5,000 schools across the Consortium have responded to a preliminary pool of test questions and performance tasks (12).

In Spring 2014, SBAC will field test items with students. According to SBAC, “The field test is expected to involve students in about 15 to 20 percent of Consortium schools [to include all of Montana’s Public Schools] and will gather the information necessary for final evaluation of item quality (12).” The schools that participate in the field test will not be able to view the results. The field test is solely a tool for SBAC to ensure each item on the test is valid and to refine the questions. Participating districts will benefit by being able to identify technology requirements, scheduling requirements, and administrative problems that may arise prior to using the actual assessment during the 2014-2015 school year. Since modeling is an integral part of the CCSSM, it will be included in both part one (the multiple choice/constructed response) and part two (the performance task portion) of the assessment. Therefore, we will analyze the sample problems for the effectiveness of assessing a student’s ability to model.

Analysis of Sample SBAC Questions
Since we are interested in how the modeling will be assessed, we will first examine the multiple choice/constructed response portion of the assessment and then look at the performance tasks for 6th through 11th grade, which are both available on the SBAC website.

Multiple Choice/Constructed Responses
Unlike the final version of the assessment, the sample test is not adaptive. It has a variety of questions from 4th Grade to 11th Grade level, including multiple choice and constructed responses. The level of challenge increases as the grade level increases. The constructed responses require students to type in their answer and show work or explain
their reasoning. For example, one of the 4th Grade problem-solving questions poses this situation:

A rectangle is 6 feet long and has a perimeter of 20 feet. What is the width of this rectangle? Explain how you solved this problem. (13)

A possible student response that would earn full credit would be $20 - 6 - 6 = 8$ ft and half of 8 ft is 4 ft, so the width is 4 ft long. This question addresses the 4th grade standard in measurement of data, which is applying the area and perimeter formulas for rectangles in real world and mathematical problems.

According to the scoring rubrics provided by the SBAC, for constructed responses points are awarded for correct answers and correct reasoning. Partial points are awarded for fulfilling one portion of the requirements. So for the above question, if students only give the answer of 4 ft, they will receive half of the possible points since they failed to explain their reasoning but correctly completed part of the problem.

In addition to situation problem solving as in the previous example, other types of questions require dragging and dropping answers into specific categories, while still other types provide an online calculator students need to use. Most of the questions are real-world problems. Figure 2 presents a sample 11th grade-level item. Figure 3 presents the grading rubric.
Tyler earns $3.00 for every e-book he sells on his website. (E-books are books that are available electronically.) He investigated the relationship between the amount spent on advertising each month and the number of e-books sold. He used this information to determine the lines of best fit shown in this graph.

![Graph: Number of E-Books Sold vs Amount Spent on Advertising Each Month]

What is the greatest amount Tyler should spend on advertising each month? Show your work or explain how you found your answer.

**Figure 2**: Sample SBAC Question for the 11th Grade (13)
Figure 3: Sample-Grading Rubric from the SBAC 11th Grade Assessment (13)

As can be seen in Figure 2, the sample question asks the student to analyze the data and the relationship between the money spent on advertising and the number of e-books sold. This item addresses the CCSSM in 8.SP.1, 8.SP.3, and 8.F.5 (See Appendix A for details.) and is categorized as a modeling and data analysis question. The problem requires students to analyze the data, draw conclusions, and interpret the results of the graph in terms of the original situation. The instructions require students to show their work or explain their reasoning; Figure 3 gives a sample student response to these requirements. The sample question shown in Figure 2 effectively begins to incorporate modeling into the multiple choice/constructed response section of the assessment since it requires many skills necessary to modeling. As may be expected, the sample multiple
choice/constructed response questions provided by SBAC on their website do not contain any open-ended modeling questions since those forms of questions are not conducive to being open ended.

**Performance Tasks**

SBAC describes the performance task as one that

…challenge[s] students to apply their knowledge and skills to respond to real-world problems. They [performance tasks] can best be described as collections of questions and activities that are coherently connected to a single theme or scenario. These activities are meant to measure capacities such as depth of understanding, research skills, and complex analysis, which cannot be adequately assessed with selected- or constructed-response items. (12)

In other words, the performance task is more conducive to open-ended modeling type problems; therefore, we will examine all of the performance tasks from 6th through 11th grade, excluding 9th and 10th grade since performance tasks are not provided for those grade levels.

Figure 4 details a sample 6th grade level problem, which deals with optimizing the dimensions of a rectangular prism. It presents a scenario of a cereal box company wanting to redesign its boxes to contain the same volume of cereal while using less cardboard.

---

**CEREAL BOXES**

A cereal company uses cereal boxes that are rectangular prisms. The boxes have the dimensions shown.
* 12 inches high
* 8 inches wide
* 2 inches deep

The managers of the company want a new size for their cereal boxes. The new boxes have to be rectangular prisms. You will evaluate one box design the company proposed. Then you will create and propose your own design for the company.

Requirements for the new boxes:
* The new boxes have to use less cardboard than the original boxes.
* The new boxes have to hold the same or a greater volume of cereal as the original boxes.

---

**Figure 4:** Sample SBAC Performance Task for the 6th Grade-Level (13)

After reading the situation, students are given questions to answer that require them to find the volume and surface area of the current cereal box and to label its dimensions.
They are then required to evaluate one box design the company proposes and to create and propose their own design. By providing the questions for students to answer, the item limits the effectiveness of the modeling problem. That is, had the performance task ended with the situation presented in Figure 2 without following it with additional prompted responses, it would have been a more effective modeling question because students would then have to come up with a way to solve the problem without prompting. However, the additional prompted responses walk the students through steps one through three and most of four in the modeling process. (Step one requires them to find the volume, step two requires them to label the dimensions, and step three requires them to find the surface area of the original cereal box.) Students are required to only finish the modeling process by finding new dimensions and explaining why their cereal box is better. On the other hand, had this section not been included, the students would be required to complete the modeling process by themselves. Students would have to determine the problem that was being asked of them mathematically, make assumptions, determine their own approach to solving the problem, and then solve the problem.

While the problem in Figure 4 effectively begins to require students to model, because of the series of prompted responses that follow it, it falls short of a complex analysis. Instead, it ends up being a series of constructed response questions that deal with the same theme. Since this is only a 6th grade-level problem, students are still in the paint-by-number steps of modeling; therefore, the item can still represent an effective assessment of the students' ability to model with mathematics. Furthermore, since this assessment is completed on a computer, the item must be designed for easy grading. Ending the performance assessment at Figure 4 would thus be extremely difficult to grade electronically. Since electronic grading limits the parameters of problem solving, the cereal box problem could be more effectively used as a classroom-modeling task. We will discuss possible improvements for this problem to be used in the classroom later in problem 2.
Figure 5 presents the 7th grade-level performance task, which involves writing expressions and calculating calories and nutrition.

![Figure 5: Sample SBAC Performance Task for 7th Grade (13)](image)

Similar to the 6th grade-level performance task, this item walks students through the modeling process by asking specific questions that enable students to solve the problem. (Students are asked to create an expression to calculate the number of calories from fat in the sample food basket and an expression to calculate the percent of total calories from protein.) The problem asks students to determine if the sample food basket meets all of the requirements but with a twist: students are required to enter different quantities in the table and use wheat in place of rice. Then students are required to explain how their new basket meets all of the requirements. This problem is a little bit more open ended than the 6th grade problem and requires students to use many different mathematical skills such as addition, subtraction, division, order of operations, and knowledge of percentages.

Figure 6 presents the 8th grade-level sample item, which is somewhat more effective at requiring modeling because it actually requires students to create a model that shows the relationship between an animal’s pulse rates and its body weight.
In this task, you will use data to create a model that shows the relationship between animal body weight and pulse rate measures. Then you will examine additional data to evaluate your model.

A study states that the relationship between an animal’s pulse rate and body weight is approximately linear. The study data are below.

Table 1. Average Body Weight and Average Pulse Rate of Seven Animals

<table>
<thead>
<tr>
<th>Animal</th>
<th>Average Body Weight (in kilograms)</th>
<th>Average Pulse Rate (in beats per minute)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cat</td>
<td>3</td>
<td>130</td>
</tr>
<tr>
<td>Goat</td>
<td>28</td>
<td>75</td>
</tr>
<tr>
<td>Sheep</td>
<td>56</td>
<td>75</td>
</tr>
<tr>
<td>Pig</td>
<td>192</td>
<td>95</td>
</tr>
<tr>
<td>Ox</td>
<td>362</td>
<td>48</td>
</tr>
<tr>
<td>Cow</td>
<td>465</td>
<td>66</td>
</tr>
<tr>
<td>Horse</td>
<td>521</td>
<td>34</td>
</tr>
</tbody>
</table>

Figure 6: Sample SBAC Performance Task for the 8th Grade (13)

In this problem, students are provided a table of data (as shown in Figure 6), along with a graph of points plotted on a Cartesian plane. Students are required to use the connected line tool to create a linear model of the data points and then give an equation for that line. They must then interpret the results. As in the performance tasks for grades 6 and 7, students are guided toward their answers; therefore, this task is not as open-ended as the most effective modeling problems are. However, the question requires more critical thinking than the previous problems do. After the students are guided through steps one through three of the modeling process, they are then required to solve the model by giving an equation for the line they created and to interpret the results according to the slope of the equation they found and make a prediction given other data. To do this, students have to recognize the slope as a rate of change (for every kilogram in body weight, the pulse decreases x amount). They are then required to predict the average pulse rate given a certain weight. The given weight would require a negative pulse rate, which is impossible, so students are required to analyze the results and explain why the
result is unreasonable. They are then required to analyze their results even further when they are given two animals with actual pulse rates and then compare the two new data points to the predicted value given with the equation they found. Thus, although the problem uses prompted responses, it also requires more critical thinking and analysis of the results.

Figure 7 presents an 11th grade-level performance task, which even more effectively requires modeling than the 8th grade-level task because this problem requires student to work with graphing functions that represent speeding fines in the state of New York and compare them to Massachusetts speeding fines.

**Figure 7: Sample SBAC Performance Task for 11th Grade** (13)
In this problem, students are required to use the results to propose a fairer fine system for speeders in New York. Students have to plot the data from the table themselves but are prompted to do so. They are then required to create an equation to calculate the Massachusetts speeding fines based on the number of miles per hour over the speed limit (between 1 mph and 10 mph over) and an additional equation based on the number of miles per hour over the speed limit greater than 10. They are then prompted to create a model using actual data for New York and compare the two fine systems. While not completely open-ended because prompted responses are used to frame the problem and how to solve it, this performance task requires students to analyze and interpret the results and then change the results into non-mathematical terms. Therefore, as can be seen in the problems provided in Figures 4-7, as the grade level increases, so does the level of challenge and difficulty.

Gradually, performance tasks provide less scaffolding and become more open ended. The test writers seem to have intentionally graduated the level of difficulty because a 6th grade-level student would have less experience with modeling than an 11th grader. Students at lower grade levels need more structures to aid in their growth as problem solvers. In addition, as long as the performance tasks are graded by computer, there are limitations on how open-ended a problem can be. On the other hand, while the sample problems for both the constructed response questions and performance tasks do not all require modeling, the questions do require students to apply mathematical concepts to real-world scenarios and to explain their reasoning. By creating a mixture of multiple choice questions and performance tasks, the SBAC assessment of the CCSSM differs from previous primarily multiple-choice state assessments. Although the problems on the SBAC assessment may not be completely open ended and thus guide students through the modeling process, the support the questions provide in the form of prompted responses is needed when first learning. Nevertheless, to be truly proficient at problem solving students must be exposed to different types of open-ended modeling problems which allow them to be creative in their approach and truly be required to think critically.
Since CCSSM and the SBAC assessment require math modeling experience, classroom instruction must address modeling. Although teachers’ curriculum should not be derived from the assessment, it is derived from the standards that the assessment represents. If students are required to think critically and solve open-ended modeling problems frequently throughout their mathematical education, then they will be adequately prepared for the new assessments. The questions remain: what should mathematical modeling look like in our classrooms, and should it be the skeleton or the muscles of math curriculum?

**MODELING IN THE CLASSROOM**

As the sample performance tasks in Figures 4-7 indicate, modeling in math classrooms should gradually become more challenging, which will then require students to think critically, make sense of problems and persevere in solving them, construct viable arguments, and use appropriate tools (stated goals of the CCSSM). Kindergarteners are not given paint, paintbrushes, and a blank canvas and expected to paint a Pablo Picasso masterpiece. They are first taught how to color in the lines. Gradually constructs are removed, making the young learners rely on their own creativity and knowledge to create their own art. So is it with implementing modeling in our classrooms. Students should be exposed to some sort of simple, structured mathematical modeling beginning in kindergarten. At the elementary grade levels, modeling can take the form of coming up with an expression that represents some real-world problem or making a bar graph representing data that the students have gathered. In order to teach modeling, teachers must scaffold their instruction, initially give lots of support, and then gradually remove the support as students develop autonomous learning strategies, thus promoting their problem solving and critical thinking skills.
Modeling in Middle School

We cannot expect students to be able to do large-scale modeling problems without adequate classroom preparation. Students must have a strong foundation of modeling in middle school so that they have a basic understanding of the process and are proficient in problem solving and writing mathematical papers by the time they enter high school. To achieve this goal, we need to provide the building blocks for modeling earlier than is currently the practice. As previously mentioned, in elementary grade levels (K-5th) modeling can be taught and practiced at a very small scale since young learners, ages 5 to 11, have generally not yet developed the ability to reason abstractly. By the time students reach middle school they are normally 11 and 12 years old so more can be expected from them due to their increased cognitive abilities. It is hard to know exactly how to integrate modeling in middle school, so it is beneficial to look at sample modeling problems intended for 6th-8th grade. First we will examine a simple problem that could be used for modeling in the 6th grade, or even as early as the 4th grade, depending on how in depth the students are required to go. A simple project to introduce modeling and writing mathematical papers would be to present students with a shortest distance problem as illustrated in Problem 1 below.

Problem 1: Shortest Distance

Instruct students to find the shortest route from the classroom to some other area within your school such as the cafeteria, gym, computer lab, or designated fire evacuation area. Students may use any desired method of measurement, but the result needs to be converted to feet. Students will have to get creative and use objects like brick lengths, tile lengths, or their actual feet, and then convert their measure into feet. Require students to come up with at least two different routes to get to the designated area, pick the optimal route, submit a written report of their findings and then present their findings to the class. (14)
In this problem, students are required to convert measurements and apply and extend previous understandings of perimeter and measurement to solve a real-world distance problem. Students will discover mathematically that the shortest distance to a destination is a straight path. While this problem is very simple, it requires students to problem solve and to use their knowledge of perimeter and measurement to write a mathematical paper. See Appendix A for full problem and additional teacher resources.

We will now examine the cereal box optimization problem introduced as a performance task for 6th grade in the SBAC assessment and discuss how it can be left open-ended so it is more effective as a modeling problem.

**Problem 2: Cereal Box Optimization**

Give students cereal boxes and measuring tapes. Instruct them to come up with new dimensions of the cereal box that minimizes the amount of cardboard used while still containing the same volume of cereal. They will have to decide if using a rectangular shape is beneficial and if not what shape would be more effective. Have students actually construct their new container and present it to the class. Instead of walking the students through all the information and giving them new dimensions to compare, leave the problem open ended. This will force students to problem solve and apply mathematical concepts to real life problems.

The goal of this problem is to minimize the surface area of the cardboard box while keeping the same volume. Problems similar to this one can be found at each mathematical level up to calculus because many real-world companies are concerned with minimizing packaging costs. While calculus is not needed in this problem, students are forced to get creative and problem solve. This problem addresses the 6th grade geometry standard which requires solving real-world and mathematical problems involving surface area and volume. Students do not need to know how to find the volume of a rectangular prism prior to this project. This modeling problem can be used as a tool
for teachers to introduce volume and surface area where students explore this topic and then teachers introduce the formulas afterwards. They can also explore the topic during the project after students have spent a sufficient amount of time problem solving.

Problem 3 presents an example of an 8th grade-level geometry problem.

**Problem 3: Gumball Cones**

Instruct the students to create a cone that will hold the most gumballs. Have the students make their cones by cutting a sector from a circle and taping the edges together. Each student in the group should make a cone that he/she believes will hold the largest number of gumballs. The students will need to keep track of the angle of the sector they cut out. The students will keep track of the number of gumballs that fit in the cone, find the volume of one gumball, and the volume of the cone. The group will combine the information they found and write a report detailing their findings. Students should be able to come up with a model that predicts the angle of the cone that holds the most gumballs given any size of gumball. (14)

This problem addresses the 8th grade geometry standard of solving real-world and mathematical problems involving the volume of cylinder, cones, and spheres. Again, this problem is a way to introduce a topic like finding the volume of a sphere as exploratory learning because it forces students to explore volume and angles of cones without having to know a formal definition or equation. When students discover ideas and formulas for themselves, especially after struggling a little bit, meaningful learning is more likely to occur.

**Modeling in the High School**

Since modeling is in the high school domain, I will focus more on high school problems. It is ideal for students to be exposed to modeling beginning in kindergarten and be given
open-ended problems throughout their schooling. It is assumed in the CCSSM that by the
time students reach high school, they should have a solid mathematical foundation and be
able to use all mathematical tools they have learned to creatively solve open-ended
modeling problems. In order for modeling to be effective, we need to choose good
modeling problems. Again, it is helpful to see a wide variety of usable modeling
problems, ranging from Algebra topics to Geometry and Probability.

Students learn skills related to science when they are required to collect and analyze their
own data. Problems that involve collecting and analyzing their own data result in
meaningful learning and often are fun for students and teachers. One problem that
requires data collection, analysis, and predictions is illustrated in problem 4 below.

**Problem 4: Soap-box Derby**

Give students a toy car—match box or hot wheel—to explore the
equation distance = rate \times time by simulating soap-box derby trials. Students
will determine the average rate a car travels by measuring distance and time
during multiple trials. Allow students to come up with their own test distances on
flat surfaces as well as and let them determine how to put the cars into motion.
Once an average rate has been determined, students can make predictions on how
long it will take to complete a race given a distance. Then the entire class can
conduct a race, with the team with the closest prediction winning. Ramps and hot-
wheel tracks can be used to assist with the trials. (14)

Problem 4 can be used in an Algebra class or a higher-level math class if we include
statistics. This problem is very simple, yet accessible. It is a meaningful modeling
problem because students are able to collect their own data, plot the data on a Cartesian
plane, fit a function to the data, and create equations that describe the relationship
between distance, rate, and time. This problem addresses standards in the Algebra
domain.
Problem 5 can be used in an upper level algebra class.

**Problem 5: Midges**

Provide the students with this background information.

**Background Information:** In 1981, two new varieties of a tiny biting insect called a midge were discovered by biologists W. L. Grogan and W. W. Wirth in the jungles of Brazil. They dubbed one kind of midge an **Apf** midge and the other an **Af** midge. The biologists found out that the **Apf** midge is a carrier of a debilitating disease that causes swelling of the brain when a human is bitten by an infected midge. Although the disease is rarely fatal, the disability caused by the swelling can be permanent. This is no insect to mess with! The other form of the midge, the **Af**, is quite harmless and a valuable pollinator. In an effort to distinguish the two varieties, the biologist took measurements on the midges they caught. The two measurements taken (in centimeters) were of wing length and antenna length.

<table>
<thead>
<tr>
<th>Af Midges</th>
<th>Wing Length (cm)</th>
<th>1.72</th>
<th>1.64</th>
<th>1.74</th>
<th>1.7</th>
<th>1.82</th>
<th>1.82</th>
<th>1.9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Antenna Length (cm)</td>
<td>1.24</td>
<td>1.38</td>
<td>1.36</td>
<td>1.4</td>
<td>1.38</td>
<td>1.48</td>
<td>1.38</td>
</tr>
<tr>
<td>Apf Midges</td>
<td>Wing Length (cm)</td>
<td>1.78</td>
<td>1.86</td>
<td>1.96</td>
<td>2</td>
<td>2</td>
<td>1.96</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Antenna Length (cm)</td>
<td>1.14</td>
<td>1.2</td>
<td>1.3</td>
<td>1.26</td>
<td>1.28</td>
<td>1.18</td>
<td></td>
</tr>
</tbody>
</table>

Have the students determine if it is possible to distinguish an **Af** midge from an **Apf** midge on the basis of wing and antenna length. Have the students write a report that describes to a naturalist in the field how to classify a midge he or she has just captured. (15)

The Midge problem is a problem from the Consortium for Mathematics and Its Applications (COMAP). Students are required to graph the data, and then find lines to the **Af** Midges and **Apf** Midges by minimizing the sum of the residuals squared. Once students have done this, they need to determine where a boundary line may possibly be to
determine the classification of the two types of midges. In this problem, students compare bivariate data graphically and interpret the results. This problem addresses standards in the Algebra, Function, and Statistics domain of the CCSSM.

Problem 6 is a higher level algebra problem.

**Problem 6: Cookie Revenue**

Give students the following information on a piece of paper: The Fresh Bakery currently sells 1000 large chocolate-chip cookies each week for $0.50 apiece. The bakery would like to increase its revenue from cookie sales by increasing this price. A survey has convinced the manager that for every $0.10 increase, the bakery would sell 70 fewer cookies each week. At what price should the cookies be sold to maximize the revenue to the bakery?” Students will need a computer, graphing calculator, or graph paper in order to graph this problem. To add more uncertainty to this problem, thus making it more difficult, tell the students that a $0.10 increase will cause 65-75 fewer cookies to be sold. Students will have to make more assumptions initially in order to solve the problem. (16)

This problem requires students to come up with a function that models the scenario, graph the function, and interpret the results. Students will be able to see the optimal price to sell the cookie occurs at the vertex of the parabola. This problem addresses standards in the Algebra and Functions domains.

**MODELING AS THE FRAMEWORK**

What would a mathematics classroom look like with these types of problems as a framework? Because modeling is best implemented as activities rather than as a concept to be taught, it is integrated throughout the high school CCSSM. The six sample problems presented are useful building blocks for modeling. All of the problems address multiple standards and require students to use various mathematical skills to solve them. Problems 5 and 6 are more open ended than problems 1 to 4. Some other examples of
open-ended mathematical problems based on real-world experiences that appropriately challenge students can be found below:

1. A doctor wants to prescribe a safe but effective dosage of a prescribed drug. How much should the doctor prescribe and how often should the drug be taken?
2. How far does a vehicle travel after the driver perceives a need to stop?
3. What should a cleaning schedule for a hotel look like and how should the cleaning resources be used? (17)

The ability to solve these types of problems should be our goal for students; however, we have to gradually develop the students' ability to solve them by starting the students off with more structure and support.

Modeling should be the framework—or the “skeleton” as opposed to the “muscles”—of the mathematics curriculum in a high school classroom. This means a modeling problem should be introduced at the beginning of a unit which would require students to explore the math concepts in order to solve the modeling problem. Teachers should provide support and instruction as problems arise. Like the skeleton of a human body, modeling provides the structure and shape of the classroom. This analogy is in contrast to the opposite analogy of modeling as the muscles of mathematics. If modeling were the muscles, students would learn all the skills needed for modeling, and then at the end be given a modeling problem that applies to all or some of these skills.

While modeling as the muscles may seem more accessible and easier to implement into current classrooms, leaving the modeling for the end of instruction is not as effective in developing problem solving skills. When modeling is the framework or bones, it creates an environment of inquiry-based learning wherein math serves the conversation. This type of environment ensures students are actively participating in their learning process, which results in meaningful learning and a better understanding of mathematical concepts.
Modeling in our classrooms benefits both students and teachers; however, barriers exist to its implementation. Both the benefits and barriers are discussed in the next section.

**Benefits**

There are many benefits to having modeling as a framework. One of the main goals of teaching math with modeling is to give students a deeper understanding of mathematics. Some other benefits relate to issues of meaningful learning, making connections, making students career and college ready, and providing flexibility for the instructor.

**Meaningful Learning**

Rather than writing down rules and solving equations, students will be interested in a modeling activity that connects mathematics to their world. This method of instruction is more likely to lead to meaningful learning. When students learn concepts through modeling, they will more likely remember the modeling problem than the equations because they actively participated in their own learning process.

**Making Connections**

Modeling shows how math concepts are interconnected, which makes it easier for students to connect one situation to another. Modeling also connects math to the “real world” and other disciplines. It reinforces concepts already learned by naturally requiring students to spiral through material since students must use all their mathematical tools to solve problems. Modeling forces students to review material and use previously learned skills constantly.

**Making Students College and Career Ready**

Regardless of whether or not a student graduates and moves on to a STEM career, a student who can model with mathematics can think critically, a skill that is needed in college level courses and in most jobs. Modeling forces students to think critically and
creatively. Students will have to use a higher order of thinking and use problem-solving skills to complete a problem. Students also become more proficient in writing and in the use of technology, which better prepares them for college and the workforce.

*Providing Teacher Flexibility*
Teachers can easily adjust the structures given to the students to solve the modeling problem. Even in the same classroom, teachers can differentiate or modify the projects to fit each student’s individual needs.

**Barriers to Implementing Modeling**
Despite the many benefits of modeling, there are some barriers to implementing modeling that should be acknowledged.

*Unfamiliar Territory*
Modeling is new territory for most teachers. This issue has been discussed in the subsection titled *How the CCSSM Differs from Previous Standards*. Few teachers have exposure to or formal training in how to implement modeling in the classroom and are therefore wary and uncomfortable teaching it. Consequently, it is hard for teachers to know how much help to give students without taking away from the problem solving experience.

*Time*
Modeling can be impractical because it is time consuming. It takes time to complete a modeling problem; most problems are longer than a chapter in a book. Students must understand the context of the problem and the skills needed to solve the problem. Some modeling problems can take more than a few class periods—and even up to weeks—to finish.
Assessment

It is difficult to assess modeling since there are no right answers in modeling, just better solutions. This difficulty poses a problem of fair grading since there are no write answers.

Resources

Completing modeling projects often requires a lot of resources and technology. Since students cannot always be expected to complete research from home, modeling has to be done at school, which poses a problem because it becomes problem of teacher and resource availability.

Finding Good Problems

It is difficult to find modeling problems that work well in the classroom given time and resource constraints. It is also difficult to find problems appropriate for every level, especially the lower grades.

CONCLUSION

The CCSSM were developed by the NGA and CCSSO in 2009, finalized, and released in 2010. To date, 44 States have adopted the CCSSM. The NGA and CCSSO intend to provide clear, consistent, and rigorous standards for mathematics so American students can compete with the ever-changing global economy. One of the major changes that occurred in the standards was the inclusion of mathematical modeling. Modeling appears in two places within the CCSSM. It is one of the eight standards for mathematical practice and one of six domains in the high school.

Modeling is often foreign to many teachers, thus difficult to implement without any training. We explored the CCSSM assessment created by the SBAC and analyzed the questions. We found that the SBAC practical exercise was more conducive to mathematical modeling and that some useful and practical elements existed within both
parts of the assessment; however, both sections fell short of being completely open-ended modeling problems. We also examined possible modeling problems that can be used from 6th grade through 11th grade.

With the implementation of the CCSSM, a paradigm shift should occur in the classrooms throughout the United States. With modeling at the forefront, teachers will be able to show the students the true beauty and art of mathematics. This will enable more students to become interested in pursuing a Math, Science, or Engineer related field.

Teaching mathematics through modeling and making it the framework of classrooms will enable students to be prepared for modern challenges. It is imperative that our future workforce is armed with the skills to be able to compete globally for jobs.
RESOURCES


APPENDIX A: Modeling Standards

HS Conceptual Category: Algebra

Seeing Structure in Expressions
A-SSE

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context.
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1 + r)^n$ as the product of $P$ and a factor not depending on $P$.

Write expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
   a. Factor a quadratic expression to reveal the zeros of the function it defines.
   b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
   c. Use the properties of exponents to transform expressions for exponential functions. For example the expression $1.15^t$ can be rewritten as $(1.15^{\frac{1}{12}})^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. For example, calculate mortgage payments.

Represent and solve equations and inequalities graphically
A-REI

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions
Interpret functions that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function $h(n)$ gives the number of person-hours it takes to assemble $n$ engines in a factory, then the positive integers would be an appropriate domain for the function.

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

   a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

   c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

   d. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
Building Functions  
F-BF

1. Write a function that describes a relationship between two quantities.
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
   b. Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.
   c. Compose functions. For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.

2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations from a variety of contexts (e.g., science, history, and culture, including those of the Montana American Indian), and translate between the two forms.

Trigonometric Functions  
F-TF

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena from a variety of contexts (e.g., science, history, and culture, including those of the Montana American Indian) with specified amplitude, frequency, and midline.

6. Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

7. Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

HS Conceptual Category: Geometry

Expressing Geometric Properties with Equations  
G-GPE
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

**Geometric Measurement and Dimension**  
**G-GMD**

3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

**Modeling with Geometry**  
**G-MD**

**Apply geometric concepts in modeling situations.**

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder; modeling a Montana American Indian tipi as a cone).

2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

**APPENDIX B: The Eight Standards for Mathematical Practice**

**Montana Standards for Mathematical Practice**

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with long-standing importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report Adding It Up: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

1. **Make sense of problems and persevere in solving them.**
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches. Building on the inherent problem-solving abilities of people over time, students can understand that mathematics is relevant when studied in a cultural context that applies to real-world situations and environments.

2. **Reason abstractly and quantitatively.**

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others.**

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They
are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions within a cultural context, including those of Montana American Indians. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. This includes solving problems within a cultural context, including those of Montana American Indians. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose

5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their
limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision.**

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure.**

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see $7 \times 8$ equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as $2 \times 7$ and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers $x$ and $y$. 
8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through \((1,2)\) with slope 3, middle school students might abstract the equation \(\frac{y - 2}{x - 1} = 3\). Noticing the regularity in the way terms cancel when expanding \((x - 1)(x + 1)\), \((x - 1)(x^2 + x + 1)\), and \((x - 1)(x^3 + x^2 + x + 1)\) might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

APPENDIX C: Teacher Resources

Problem 1: Shortest Distance

Authors: Nancy Maple, Ray Shannon, Christine Bolte, Kathy Padgett, Jehan Khan

Materials: Data sheet, measuring devices are optional.

Goals: To introduce the students to mathematical modeling problems, write-ups, and presentations through a simple distance problem.

Time Required: About two days. The problem could first be described the last 10 minutes of class on Thursday. The entire class time on Friday could be used to measure. On Monday, the write-ups would be due, and the presentations would be given. Of course, the project could also be assigned as an out of class assignment, provided the students are able to meet before or after school.

Background: Prior knowledge of measuring distances.

Setting: Finding the shortest path between two points or the shortest route through many points is a problem faced by all sorts of people and companies. Some diverse examples include taxi drivers, mail carriers, and companies which process information through computer networks.

Problem: Find the shortest route from the classroom to the pay phone. Students may use any desired method of measurement, but the final result must be converted to feet. The handout may be used to record the data for two strategies, making sure to note any
assumptions. Each group will submit a written report of their findings, and will present their results to the class.

**Evaluation:** One grade will be given for the entire group. Point values will be assigned for the data sheet, written report, and presentation. Example: data sheet/report, 10 points and presentation, five points. Bonus points may be awarded to the group finding the shortest route.

**Extensions:** The problem may be adapted to various school settings. It may be more feasible for the students to find the shortest distance to the restrooms, drinking fountain, Coke machine, nurse’s office, candy machine, etc. Restrictions may be included, such as not allowing right turns. Another extension is to consider a more complex problem such as finding the shortest route through an entire day’s schedule. Another twist to the problem is to find the quickest path between two points or through a route. The shortest and quickest are sometimes the same, but not always, as experience with crowded hallways and roadways attests.

**Teacher Notes:** In groups of three, one student could record the data while the other two measure the distances. Encourage students to be creative in their measurements by letting them devise their own method (using their feet, floor tiles, rulers, cinder blocks). If time permits, more than one strategy may be used. Be sure the expectations for the report and presentation are clear to the students before beginning the project.
Write-up – must include a description of the problem and how it was approached, a list of assumptions, calculations, a conclusion, and the data sheet.

Presentation – should discuss everything in the report.

Students may have difficulty recognizing the assumptions they make while measuring. Overlooked assumptions might include stating the exact starting point (teacher’s desk, doorway?) and final destination (which pay phone?), consistent measurement (all floor tiles are the same size, a student’s shoe measures one foot), route traveled (center of hallway, side of hallway?), not allowed to use Coke machines in teachers’ lounge, and so forth.

Sample Report for Shortest Distance Problem

Project Description: Our problem was to find the shortest route from our classroom to the pay phone. Since our group had no available measuring devices, we decided to count the number of floor tiles along our routes. Our first path was to turn left outside the room and walk to the end of the hall, and then turn right and walk to the end of that hall. We then turned left and stopped at the phone closest to us. Our second strategy was to turn left outside the room, turn right down the first hallway and go to the end, turn left and follow that hall to the end, turn right and follow that hall to the end, and then finally turn left and stop at the closest phone. We recorded our data on the data sheet. Finally, we measured the length of one tile and converted the number of tiles to feet in order to see which path was the shortest.

Assumptions: The following is a list of assumptions we made during the project:
1. The starting point was the doorway of our classroom.
2. The route would be traveled down the middle of the hallways.
3. Each floor tile has the same length of one foot.
4. The final destination was in front of the phone closest to our classroom.

Calculations: We assumed that each floor tile represents one foot.
- Strategy 1 - 376 floor tiles = 376 feet.
- Strategy 2 - 402 floor tiles = 402 feet.

Conclusion: Based on our measurements, we have concluded that our first strategy was the best, and that this route is 376 feet.
Shortest Distance

Student Lab Sheet

**Problem:** Find the shortest route from the origin to the final destination. You may use any desired method of measurement, but the result must be converted to feet. This handout may be used to record the data for each strategy. Be sure to note any assumptions you made while measuring. Each group will submit a written report of their findings, and will present their results to the class.

**Origin___________________________**  **Final**
**Destination___________________________**

<table>
<thead>
<tr>
<th>Team Members:</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Method of Measurement</th>
<th>Brief Path Description</th>
<th>Distance (feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Strategy 1</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Strategy 2</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Assumptions:

Problem 3: Gumball Cones
Authors: Ben Huntington, Vicki Miller, Barry Nielsen, Khris Willis, Susie Shafer, Paula Lincoln, and Debbie Wilch

Materials: Graph paper, note paper, pencil/pen, protractor, compass, calculator, included student sheet, math book with formulas, measuring devices, scissors, construction paper, tape/glue, large number of gumballs (spheres) of varying sizes.

Goals: Students will make a judgment as to what shape of cone will hold the maximum number of gumballs.

Time: Two consecutive days and the evenings to work on calculations and report. Due the third consecutive day.

Background: Students will need to know how to work cooperatively, how to find volume, how to estimate, how to evaluate expressions, how to measure, how to use a compass, how to use a protractor.

Setting: Groups of students are going to try to find a cone which will hold the most candy. Candy will be spherically shaped (i.e. gumballs). Students will find the cone that holds the most candy--level or heaped. Students will make their cones by cutting a sector from a circle and taping the edges together.

Problems: The student sheet states three specific problems.

Extensions: The use of non-spherically shaped objects will enhance this cool lesson and give you goose bumps all over. Increase the size of the circle (11x17 paper or butcher paper) and see how close their predictions are at a larger scale. Try baseballs or tennis balls and see how the predictions hold up.

Teacher Notes: A good opener to this project might be to give the students a sheet of typing paper and have them compare the volume of cylinders formed rolling the paper lengthwise versus widthwise. Students may have the misconception that the same size paper will construct
the same size volume. Don't offer any suggestions or tools unless the students specifically ask for them. You may always substitute inedible objects (marbles, etc.) for the candies if that works better in your class. This project will lend itself to an oral group presentation rather than a written report, if so desired.

<table>
<thead>
<tr>
<th>Cone Gumball Handout</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Each group must hand in a report solving these problems.</strong> The report must contain the following: cover page; list of any assumptions you make; calculations and formulas; data collection pages; explanations and justifications of your solution methods; your conclusions; sources of possible error; and your diagrams and models.</td>
</tr>
</tbody>
</table>

**Problem A:**
You can form a cone by cutting a sector out of a circle and taping the edges together. Each member of your group should make a cone which he/she believes will hold the largest number of gumballs: filling to level and heaping the top. Keep track of the angle of the sector you cut out. Next, find the volume of one gumball and the volume of your cone. Combine the information on the group data collection sheet. All cones need to be made from the circles provided.

**Problem B:**
Each member of the group should repeat the process from Problem A with a different size gumball, but all members need to use the same size. Enter information on the group data collection sheet. All cones need to be made from the circles provided.

**Problem C:**
Your group will be given gumballs of a new size. Predict an angle θ, and construct the cones that you hope will hold the most gumballs.
# Gumball Cone Student Page
## Data Collection Sheet

Name___________________

<table>
<thead>
<tr>
<th>angle θ of cut out sector</th>
<th>Volume of gumball</th>
<th>Volume of cone</th>
<th># filled level w/ top</th>
<th># filled when heaped</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Student 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Student 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Student 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Student 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

## Group Discussion:
1. What angle θ would hold the largest number of candies – level and heaped?

2. Find a formula so that if you know the angle θ, you can find the number of candies that can be held.
3. Would you have to make any adjustments for other larger or smaller shapes?
4. Repeat exercise with another size gumball.
Problem 4: Soap Box Derby

Authors: Cindy Boyd, Anthony Ice, Phyllis Slayton, Barb Steinbrunner

Materials: Per group: toy car (Match Box or Hot Wheels type), meter stick or tape measure, stopwatch, race track pieces are helpful to guide cars and build launch ramps

Goal: Students will form a better understanding of the equation distance=rate*time while experimenting, solving for different variables in the equation, and making predictions from data collected about a specific car.

Background: Students need to know and be able to apply the equation d=r*t, as well as to chart and interpret data.

Setting: Students are to pair off (or form groups of three or four) and choose a toy car to simulate the soap-box derby racing trials. Distances are to be measured off, and car travel times recorded. With this information, students will determine the average rate the car travels. Using this same rate (as best as possible), students are to predict how long it will take their car to travel a specific distance, which is the length of the race.

Problem: Get-Up-'N'-Go Inc. is preparing to enter a car in the annual soap-box derby. The Get-Up-'N'-Go race team continues to hear rumors of the speeds of their competitors and are beginning to worry. This team cannot afford any expensive equipment but wants to know how the speed of their car compares. Perform some test runs and help the Get-Up-'N'-Go team determine their average speed. Once their average speed is determined, use this to predict how long it will take the Get-Up-'N'-Go race team to complete the race.

Teacher Notes: Allow students to determine their own test distances on flat surfaces, as well as how the cars are to be put in motion. Students are striving to achieve a relatively constant, measurable speed, so the method of starting the cars should always remain the same. Some examples are: giving the car a beginning push with a finger, starting off on a decline or ramp preceding the flat distance measurement, or using a plunger or motorized device. Having students use toy race track pieces will help keep the cars going in a straight path, allowing for more accurate measurement. Also, race track ramps could be used to begin the car movement, rather than building ramps of books or other materials. When using ramps to initiate car movement, the possible distance to cover is limited, so a maximum distance of a yard or meter (depending on the measurement system being used) is recommended. It is beneficial to choose cars that roll well enough to cover longer distances. If desired, students could be allowed to use electric race cars on a track. The length of the soap-box derby is determined by the teacher, should not exceed the meter or yard length (unless using motorized cars), and is announced briefly before the testing of the predictions. Students are to calculate and announce the prediction, then individually test their car. It may be beneficial to allow up to three tests.
at this distance. If predictions are not coinciding with the actual race times, have groups, or as a class, discuss what could be influencing the actual race times as compared to the predictions (such as friction slowing the cars on the longer distances or having less effect over the shorter distances).

**Extensions:** If students are using motorized cars, have the students predict the winner from the various groups and simulate the soap-box derby. Perhaps a prize could be given to the winner or the most accurately predicted race performance. Other objects, such as tennis balls, could also be used as test items, rather than the toy cars. Students could estimate the speeds of passing cars by marking off a given distance and timing the passing of the vehicles. Can they catch anyone speeding?

---

**Soap Box Derby Handout**

Get-Up-'N'-Go Inc. is preparing to enter a car in the annual soap-box derby. The Get-Up-'N'-Go race team continues to hear rumors of the speeds of their competitors and is beginning to worry. This team cannot afford any expensive equipment but want to know how the speed of their car compares. Choose your car, name it, and perform some test runs to help the Get-Up-'N'-Go team determine their average speed. Once the average speed is determined, use this to predict how long it will take the Get-Up-'N'-Go race team to complete the race. (The distance and location of the race are secret and will be revealed only moments before the start of the race.)

Accuracy is of utmost importance, and Get-Up-'N'-Go Inc. wants written justification of your prediction of the travel time necessary for the car to finish the race. Submit a report carefully detailing any assumptions made, data collected, the method(s) used in calculating the average speed of the car, the average speed, and the method by which the race length prediction will be made once the distance is revealed. Also include possible sources of error for your prediction.
Problem 4: The Midge Problem

Sample Student Solution

The students noticed that both data sets overlapped, so knowing only one of the measures did little to distinguish the midges.

Of course, the first thing to do is graph the data, as shown in Figure 1. From the graph, we can see that there is a clear region between the two data sets. Where should the boundary be placed to most accurately distinguish the two species?

![Figure 1: Graph of the Midge Data.](image)

The most common solution involved first fitting a least squares or median-median line to each data set, using Wing Length as the independent variable and Antenna Length as the dependent variable. The least-squares linear fit for Af midges is $y = 0.479x + 0.549$ while for the Apf midges it is $y = 0.588x + 0.151$. (Figure 2.)

![Figure 2: Linear Least-Squares Line to Each Data Set.](image)

Since these two lines pass through the two data sets, it seemed reasonable that the midline between them would be a good boundary. To find the line that bisects the region
between these two lines, simply “average” the two lines. The boundary determined in this fashion is \( y = 0.5185x + 0.350 \). Any midge below this line was to be considered an Apf midge and destroyed, while any midge above the line was to be considered an Af midge and saved. How does it look to you? (Figure 3.)

Most students felt that this line was not a good boundary, since it seems to misclassify an Af midge as an Apf midge. Almost all groups eventually realized another approach needed to be used. However, one group steadfastly argued that this misclassification was a reasonable price to pay for added safety. They would rather say a safe midge was dangerous, and erroneously kill it, than to say a dangerous midge was safe. This line, while clearly not a good choice if your interest is accuracy, is a good line to use if your interest is safety.

**Problem 5: Cookie Revenue**

The Fresh Bakery currently sells 1000 large chocolate-chip cookies each week for $0.50 apiece. The bakery would like to increase its revenue from cookie sales by increasing this price. A survey convinced the manager that for every $0.10 increase, the bakery would sell 70 fewer cookies each week. At what price should the cookies be sold to maximize the revenue to the bakery?

**Sample Solution:**

The revenue, \( R \), is the product of the price charged, \( p \), and the number of cookies sold, \( c \). So, \( R = pc \). The number of cookies sold is a linear function of price, since for each $0.10 increase in price, the demand is reduced by 70 cookies. The slope of the line is

\[
\frac{\text{number of cookies}}{\text{price increase}} = \frac{-70}{0.10} = -700.
\]

Since the bakery presently sells 1000 cookies at $0.50 each, the point \((0, 1000)\) is on the line. So

\[
0 - 700(p - 0.50) + 1000 = 700p + 1000.
\]

The revenue, then, is given by

\[
R(p) = -700p^2 + 700p + 1000.
\]

This is a quadratic function. The optimum price is represented by the coordinates of the vertex of this parabola (Figure 1).

We expected that students would find the vertex in one of these ways:

1. Recall from Algebra the location of the vertex of a quadratic: \( x = -\frac{b}{2a} \), \( y = -\frac{b^2}{4a} \).
   - Or in this case, \( x = -\frac{700}{2(-700)} = 0.50 \).
   - \( y = -\frac{700^2}{4(-700)} = 1000 \).

2. Remember the geometry of a quadratic function with the vertex as the midpoint of the zeros, and realize the given quadratic is decreased, so the zeros are easily found at \( p = 0 \) and \( p = \frac{1000}{700} \).
   - The vertex is at \( (0.50, 1000) \).

3. Graph the function if you are not confident and ask the numerical maximum, which is reported as \( p = 0.50 \).

Regardless of the technique, we expected students to report the optimum price as \( p = 0.50 \). We expected several students to consider the optimum prices at $0.60 or $0.40, since these are more practical values and the difference in revenue is negligible. There were some concerns about the increases occurring in $0.10 increments. We also thought we might find some students adjusting the price so that, with the ideal, the total would be exactly $1.00.
Additional Modeling Problems

The College Fund Savings Problem

**Authors:** Barbara Carstensen, Deon Hickey, Tom Dahlquist, John McMahan, John Roberts, and Margaret Sanders

**Materials:** Access to computer spreadsheets.

**Goals:** This project should show the student the power of compound interest. It will allow the student to set up a table to model the answer and suggest patterns for a geometric sequence and series.

**Time Required:** Five days. The student will need to do research for some of the information. Parts of three class days should be allowed to: present the problem, monitor student progress, present solutions.

**Background:** A student should have knowledge of exponents and calculating simple interest. A student should be competent in the use of spreadsheets such as Lotus 123 or Excel.

**Setting:** Given a fixed interest rate, the student will find the amount of money to invest each year to raise the amount of money needed for four years of college. The student will do this without annuity formula.

**Problem:** Imagine that your parents were transported forward in time to the present day (Back to the Future I, II, or III?), and the only thing your parents found out was the yearly cost of a college education at any American school. Suppose your parents then decided when you were born that they would invest a certain amount of money each year for your college education. Assume that the average rate of interest for the investment is 8% annual interest rate. How much money should your parents invest each year to pay for your college education? Use the spreadsheet and a trial and error method to find solution.

1. Each group will choose a college to attend. The group needs to find the estimated cost to attend the college for four years. The groups will need to calculate how much the “parents” will have had to invest each year since your birth to finance the four years of college.
2. Each group will construct a chart using a spreadsheet to indicate the growth of the college fund from year 0 to year 21. The following headings should be on the chart.

<table>
<thead>
<tr>
<th>Year</th>
<th>Beginning Balance</th>
<th>Interest</th>
<th>Amount Invested by Year</th>
<th>Ending Balance</th>
<th>Interest Rate</th>
<th>Yearly College Cost</th>
</tr>
</thead>
</table>

3. Each group will construct a graph based on the year and the ending balance on the spreadsheet.
4. Each group will give an oral presentation to the class about their findings and what assumptions they made.
5. After class discussion, each individual will write a paper about their impressions on the project. The paper should include the student’s thoughts about possible extensions to the project.

**Sample Problem:** Demonstrate in class how interest and total value of the investment can be tracked for a few steps.

**Evaluation:**
1) Chart (20%).
2) Graph (20%).
3) Neatness (15%)
4) Organization (15%).
5) Oral presentation (15%).
6) Follow-up paper (15%).

**Extensions:**
1) What if the interest is compounded at a rate other than yearly?
2) What about investing in stocks?
3) Can you write a formula to predict what it will take to raise a certain amount of money?
4) What if the interest rates vary and are locked in for periods of time?
5) What if the parents chose to invest less each year at the beginning of the fund and more later?
6) How can you account for inflation in your calculations?
7) What about the students? How can they plan for their children's education not knowing the exact cost?

**Teachers Notes:**
The students should be aware that they will have to seek the information to solve the problem from outside sources (banks, library, investment services, guidance counselor, college chosen). A representative of a local investment firm may be a good follow-up speaker.
The College Fund Savings Problem
Student Work Sheet

Setting: Given a fixed interest rate the student will find the amount of money to invest each year to raise the amount of money needed for four years of college.

Problem: Imagine that your parents were transported forward in time to the present day (Back to the Future I, II, or III?) and the only thing your parents found out was the yearly cost of a college education at any American school. Suppose your parents then decided when you were born that they would invest a certain amount of money each year for your college education. Assume that the average rate of interest for the investment is 8% annual interest rate. How much money should your parents invest each year to pay for your college education?

1) Each group will choose a college to attend. The group needs to find the estimated cost to attend the college for four years. The groups will need to calculate how much the "parents" will have had to invest each year since the A child’s birth to finance the four years of college.

2) Each group will construct a chart using a spreadsheet to indicate the growth of the college fund from year 0 to year 21. The following headings should be on the chart.

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<th>Interest</th>
<th>Amount invested by year</th>
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<th>Yearly College Cost</th>
</tr>
</thead>
</table>

3) Each group will construct a graph based on the year and the ending balance on the spreadsheet.

4) Each group will give an oral presentation to the class about their findings and what assumptions they made.

5) After class discussion, each individual will write a paper about their impressions on the project. The paper should include the student’s thoughts about possible extensions to the project.

Evaluation:

1) Chart (20%).
2) Graph (20%).
3) Neatness (15%)
4) Organization (15%).
5) Oral presentation (15%).
6) Follow-up paper (15%).
Bungee Barbie & Kamikaze Ken Problem

**Materials:** Doll action figures, graph paper, meter sticks, calculators, two pound box of uniform rubber bands. (A project extension requires a scale for weighing figures.) Ask students to bring in action figures and/or dolls.

**Goals:** Students will work in teams, gather and graph data, fit a line to data or determine equations using data, make predictions from the model and gain an appreciation for simulations.

**Time required:** Two to five class periods.

**Background:** Students should know how to plot points on a graph and determine equations of lines.

**Setting:** Team members have been hired to work for the Acme Daredevil Entertainment Company. This company provides rock climbing, sky diving, "extreme skiing", and cliff diving adventures to the public. However, to keep up with the market, the board decided to add bungee jumping to its list of offerings. As part of the first assignment, the board decided that the teams should undertake the task of working out the details of the new venture. The company has several sites planned for bungee jumping and each site is at different height.

**Problem:** Initially each group works with just one action figure (doll). The task is to determine the ultimate length, or the number of rubber bands that can be used with your action figure at any given height and not cause any type of injury or fatality--but that allows your action figure to come as close to the floor as possible (for maximum thrills).

**Solution Requirements:**

1. Teams will need to be able to predict the length of rope for any given height to them by the teacher. Predictions will be checked through actual experimentation.
2. List any assumptions that were made.
3. Include graphs and tables.
4. Present methods of solution and justifications of conclusions.
5. Include possible sources of error.

**Evaluation:** Students will work in groups and presentations should include graphs, tables, equations, assumptions, error bounds, conclusions, and predictions. Written and oral reports may be used in addition to the actual jump length prediction results.

**Extensions:** This project could be repeated using rubber bands with different strength coefficients or other types of elastic materials. Dolls of differing weights can be used in discovering what effect that has on the band. Problems in the text that refer to Hooke's Law can be modeled by the students.
**Teacher Notes:** Students should make at least three trials when dropping Barbie from a specified height and use the average. Test drop several times to practice taking readings. Drop Barbie with 1 band and measure; drop Barbie with 2 bands and measure; continue doing this until you are using at least 6 rubber bands. Data can be entered on a graphing calculator or graph paper. Use the statistics features of a TI-82 to fit a curve, or the table function to make predictions from the calculator or by extending the line and reading the results off the paper. Consider dropping from a balcony, stadium, gym bleacher or platform ladder. Do not tell where test jump will be made.

If weight is being considered, then use this to find the stretch coefficient: 

\[
\frac{\text{Amount of stretch}}{\text{weight}} = \text{stretch factor}
\]

Buy plenty of rubber bands (2lbs box from an office supply store; size should be about 4in in length un-stretched). Be aware that after several uses the rubber bands will permanently deform or stretch and that this may affect the problem. Let students discover and cope with this complication in any reasonable way (perhaps using new rubber bands frequently or for the final test jump, or pre-stretching rubber bands).

**Sample Solution**

**Problem:** Develop a model to predict the number of rubber bands needed to drop a Barbie of our choice when given a height.

**Assumptions:**
1. The rubber bands are identical to one another.
2. Height and number of rubber bands are related linearly.
3. Barbie will start from a standing position and will free fall head first.

**Procedure:** Collect data by measuring the height of a Barbie’s fall with various numbers of rubber bands. Repeat each trial three times and average the results.

**Data:** Height in cm.

<table>
<thead>
<tr>
<th>Number of Rubber Bands</th>
<th>Jump1</th>
<th>Jump2</th>
<th>Jump3</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>31</td>
<td>33</td>
<td>30</td>
<td>31</td>
</tr>
<tr>
<td>2</td>
<td>52</td>
<td>55</td>
<td>51</td>
<td>53</td>
</tr>
<tr>
<td>3</td>
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**Model:**

\[
X = \# \text{ of rubber bands}; \quad Y = \text{cliff height (CM)}
\]

\[
Y = 21.2X + 9.8 \quad \text{or} \quad X = (Y - 9.8)/21.2
\]

Equation obtained by linear regression on TI-82 calculator and has a correlation coefficient of .9997.

**Prediction:** We were given a test height of 595cm. Our model gives a prediction of 27.6 rubber bands. To be on the safe side we will use 27 rubber bands. During the actual test we came within about 16cm of the floor.

**Reflection:** We could have used another one (or possibly two) rubber bands, and still
have had a safe jump. Possible sources of error include consistency of rubber bands, linearity assumption at extreme heights when rubber bands are stretched to a greater length, and the permanent deformation of the bands after the first jump should always use fresh bands for data collection trials and the test jump.

**Additional Websites with Modeling Problems/Resources**

http://www.comap.com/highschool/contests/
http://www.mathmodels.org/problems/
http://www.indiana.edu/~hmathmod/projects.html