Spring 5-13-2017

Mirror symmetry in black holes, elementary particles, and mathematics

Mark Romano
Carroll College, Helena, MT

Follow this and additional works at: https://scholars.carroll.edu/mathengcompsci_theses

Part of the Applied Mathematics Commons, Astrophysics and Astronomy Commons, Elementary Particles and Fields and String Theory Commons, and the Mathematics Commons

Recommended Citation
Romano, Mark, "Mirror symmetry in black holes, elementary particles, and mathematics" (2017). Mathematics, Engineering and Computer Science Undergraduate Theses. 2.
https://scholars.carroll.edu/mathengcompsci_theses/2

This Thesis is brought to you for free and open access by the Mathematics, Engineering and Computer Science at Carroll Scholars. It has been accepted for inclusion in Mathematics, Engineering and Computer Science Undergraduate Theses by an authorized administrator of Carroll Scholars. For more information, please contact tkratz@carroll.edu.
This thesis for honors recognition has been approved for the Department of Mathematics, Computer Science, and Engineering.

Director
4/28/17
Date

Reader
4/27/17
Date

Reader
4/27/17
Date
Mirror symmetry in black holes, elementary particles, and mathematics

Mark Romano

May 1, 2017
Abstract

Mirror symmetry is the study of two seemingly different objects that are conjectured to be the same. That is, we could think of them as mirror images of each other. For example, string theorists conjecture that there exists such a duality between black holes and elementary particles. This thesis discusses this example from string theory, and demonstrates some mathematical mirror symmetry examples involving polynomials and groups. Given a pair made up of a polynomial and a group, we create a vector space. The vector spaces created by the dual pair always turn out to be equivalent! Progress in this field has been made by breaking polynomials into smaller pieces (atomic-types), called Fermats, chains, and loops. In this thesis I will give a formula and complete description of the vector spaces associated to Fermat and loop polynomials with repeated exponents and using their maximal symmetry groups. Finally, explanations on general observations on the occurrence of vector spaces with various dimensions based on computations will be explored, and explanations for various methods and formulas obtained during my study of these objects will be given.
## Contents

1 Introduction ........................................ 5
   1.1 String theory, duality of elementary particles, and black holes .... 7
   1.2 Understanding mirror symmetry .................................. 8

2 Landau-Ginzburg Symmetry ............................ 10
   2.1 Calculating the A-model as a vector space .................. 10
   2.2 New computations and proofs .................................. 13

3 Future work ........................................... 24
List of Figures

1  Diagram of the duality of multiplication and logarithmic addition. . . . 5
2  Group tables for $V_4$ and $R_4$ where $R_4$ is on the left and $V_4$ is on the right. 6
3  A 2D cross section of a 6 dimensional Calabi-Yau manifold. Image from [1] 9
4  Diagram of global mirror symmetry. . . . . . . . . . . . . . . . . . . . 10
5  Plot of the function $y = |x|$. . . . . . . . . . . . . . . . . . . . . . . . . . 11

List of Tables

1  All two and three variable Fermat, loop, and chain polynomial with exponents up to eight yielding four dimensional vector spaces. . . . 21
1 Introduction

Mathematics and physics have long had a close relationship which has furthered both fields [8]. In the last hundred years however, the increasing abstractness of mathematics has led to a disconnect between the two fields [8]. Historically there have been many relationships between physics and math, but we will look at two examples which illustrate the concepts of isomorphisms and duality.

In [8], Zaslow discusses the duality of multiplication and logarithmic addition. For example, multiplying numbers is actually dual to adding logarithms. First we translate the two numbers we wish to multiply into logarithms. We then add them and finally convert them back to a number. By doing this we arrive at the same solution as if we just multiplied them. This illustrated a duality. Although the two processes, multiplication and logarithmic addition appear very different, they actually result in the same solution. Figure 1 shows how this process is done.

\[
(a, b) \quad \xrightarrow{\text{Convert to logarithms}} \quad (\log(a), \log(b))
\]

\[
(a, b) \quad \xrightarrow{\text{Multiplication}} \quad a \times b \quad \xleftarrow{\text{Convert from logarithms}} \quad \log(a) + \log(b)
\]

Figure 1: Diagram of the duality of multiplication and logarithmic addition.

Next we look at the study of categories in mathematics. These are groups of objects and the maps which link them. They can be sets, vector spaces, groups, or other mathematical objects. A well known example from algebra is groups for four
elements, $R_4$ and $V_4$. The group $R_4$ is also known as $Z_4$ and can be thought of as the group generated by rotations of a square. The group $V_4$ is the Klein four group and can be thought of as vertical and horizontal flips of a square. The group tables can be seen in Figure 2. If we did not look at the group table for these, we might assume that they are isomorphic as they both are groups of four elements. However, they have additional structure that is revealed in the group table. In $V_4$, all elements are order two, which means that multiplying them by themselves once results in the identity. If we look at $R_4$ however, there is only one element of order two. Although these groups appear similar, the difference in their inner structure excludes them from being isomorphic.

![Figure 2: Group tables for $V_4$ and $R_4$ where $R_4$ is on the left and $V_4$ is on the right.](image)

A pertinent example of duality from string theory, given in [8], is the energies of strings. Each string has a winding number and momentum which is inherent to that string. We will summarize this example here. First we can examine the energy of a string that exists on a circle of radius $R$. Strings on this circle with a winding number of zero, are low energy strings. We can then construct a string on a circle with radius $\frac{1}{R}$. The strings on this circle with a momentum of zero are dual to those on the circle of radius $R$ with winding number of zero. This shows that the momentum of strings is dual to the winding number of strings. Although the strings in these two “worlds” look completely different, they are in fact dual to one another.

String theory has not yet been proven, but if it does turn out to be correct then it will be the link between quantum theory and Einstein’s general theory of relativity. In this case then, the study of duality will be crucial to unifying quantum theory and
Einstein’s general theory of relativity. In the rest of this paper we will explore topics of mirror symmetry could be used to explain the underlying physics.

1.1 String theory, duality of elementary particles, and black holes

In string theory, it is theorized that there is only one substance and all matter is different vibrations of this one substance. If we consider this in two dimensions we can visualize a sinusoidal curve. As we increase the energy of the system, we can increase the frequency of the wave. In this example, different forms of matter would be seen as sinusoidal waves of different frequencies. The reason that we do not see the infinite forms of matter then is because we cannot introduce enough energy to the system. The study of string theory deals with these ideas in higher dimensions.

In [6], Susskind discusses the new theory of black hole evaporation into strings. A black hole is a singularity with infinite density, but a finite amount of mass. A singularity is classified as a black hole when it has enough mass and gravity that not even light can escape it. To understand how black holes become strings we examine what happens as the gravitational constant is allowed to approach zero. When it approaches zero, as is theorized to happen as a black hole grows, then at some point the black hole must become a string. This occurs because the gravitational force compresses all the matter to a uniform singularity. The black hole can then either become a loose collection of multiple strings, or it can become one string. Based on the black hole’s entropy however, it is more stable if it becomes one string. If this is true, then from the point of view of string theory, the black hole will resemble a particle as it is just one string.

In [2] Coyne and Cheng discuss a new theory of elementary particles. We will offer a brief summary of their results here. They offer the explanation that all elementary particles are actually stabilized black holes. They suggest that there are two types
1.2 Understanding mirror symmetry

of gravity that are observed in the universe. The first is strong gravity which we see around black holes. The second is the weaker gravity that we see everywhere else that holds the universe together. Thus, it may be that this gravity is actually just leakage of strong gravity from elementary particles. Their reasoning is that the strong gravity that black holes exhibit is still present in all elementary particles, but it is shielded in other dimensions which we cannot see. The only gravity that we see from elementary particles is the gravity that leaks from this shielding: weak gravity. As humans only observe four dimension (the three spacial dimensions and time) we cannot see the effect of strong gravity in our universe. The only direct observation we can see of this strong gravity then is around black holes which have not yet stabilized to elementary particles.

If this conjecture is true, then elementary particles and black holes are dual to one another. They in no way appear similar, but could one day be proven to be structurally equivalent. Mirror symmetry is based on this idea where two objects seem completely different, but they are structurally the same. Based on the papers cited above, there may in fact be no fundamental difference between black holes and elementary particles. If this is true, then the mathematics behind mirror symmetry may one day be used in a theory that unifies Einstein’s physics and quantum physics.

1.2 Understanding mirror symmetry

Mirror symmetry is a blanket term for the relationship between two different models of duality. The first such instance was proposed after string theory was first conjectured. First researchers observed a duality in string propagation when examining circles of radii $R$ and $\frac{1}{R}$, an example which was discussed earlier. Then Calabi-Yau manifolds were discovered to be a geometric representation of string propagation. Calabi-Yau manifolds are multi-dimensional geometric objects which always occur in pairs. A cross section of a six dimensional Calabi-Yau manifold can be seen in Figure 3. The
1.2 Understanding mirror symmetry

Romano

Figure 3: A 2D cross section of a 6 dimensional Calabi-Yau manifold. Image from [1]

observation that these occur in pairs, but look completely different is what began further study in this field.

The first explorations into mirror symmetry were in the Calabi-Yau manifolds. These can be represented by polynomials and in some cases it is possible to prove that the pair of dual manifolds are equal. The problem is that this is very difficult in Calabi-Yau symmetry. There is, however, another version of mirror symmetry which is theorized to be easier to work in: Landau-Ginzburg mirror symmetry. In Calabi-Yau symmetry, the weights of all the variables must sum to one, but in Landau-Ginsberg symmetry this is not required.

In 1991, Edward Witten conjectured that these two types of mirror symmetry were actually dual to one another [3]. This conjecture has been partially proven as it has been proven for some cases. This could mean that we can work in the possibly easier Landau-Ginzburg symmetry to answer the same questions Calabi-Yau symmetry tried to answer. Figure 4 is a diagram of global mirror symmetry and shows the relationship between the two fields.
2 Landau-Ginzburg Symmetry

2.1 Calculating the A-model as a vector space

The output of Landau-Ginzburg mirror symmetry is a vector space. In this section we will show the steps to create this vector space beginning with a polynomial and group and performing the calculations to produce the vector space.

In order for a polynomial to exhibit Landau-Ginzburg symmetry, the polynomial must be a nondegenerate, quasihomologous polynomial. A polynomial is nondegenerate if it has an isolated singularity at the origin. This means that the derivative does not exist at this point. For example, in two dimensional space, we can think about the function $y = |x|$.

Figure 5 shows the graph of $y = |x|$ on the domain $[-10, 10]$. As we can see, the right and left limits of the slope do not agree at the origin, which means that the derivative is not defined there. We can then say that the function has a singularity at the origin. This idea applies at higher dimensions.

Next the polynomial must be quasihomologous. A polynomial is quasihomologous if there exist weights $q_1, q_2, ..., q_n$ such that for each monomial $\prod x_i^{a_i}$, $\sum a_i q_i = 1$.

Example 2.1. We can have the polynomial $W = x^2y + y^4 + z^5$ which is quasihomol-
2.1 Calculating the A-model as a vector space

We define three atomic types of polynomials. These are

- Fermat polynomials of form $x_1^n$

- Loop polynomials of form $x_1^{n_1} x_2 + x_2^{n_2} x_3 + x_3^{n_3} x_4 + \ldots + x_k^{n_k} x_1$

- Chain polynomials of form $x_1^{n_1} x_2 + x_2^{n_2} x_3 + x_3^{n_3} x_4 + \ldots + x_{k-1}^{n_{k-1}} x_k + x_k^{n_k}$

**Theorem 2.2. ([5])** Any nondegenerate, quasihomologous polynomial is the sum of Fermat, loop, and chain polynomials.

Once we have a polynomial $W$ which satisfies these requirements, we must define a symmetry group for $W$. The symmetry group is a group which fixes the monomials under rotations. For example if $W = x^3 + y^3$, the the first entry of our group elements is chosen from those rotations that fix the first monomial $x^3$. Therefore, we must have $(e^{2\pi i g_1} x)^3 = x^3$, where $g_1$ (the first entry of group element) could be $0, \frac{1}{3},$ or $\frac{2}{3}$. Likewise, the second entry of the group elements must “fix” the second monomial and has the same options of $0, \frac{1}{3},$ or $\frac{2}{3}$. Therefore, the group, with operation of rotations,
2.1 Calculating the A-model as a vector space

Romano

is made up of all combinations of these two entries and has nine elements. This group is

\[ G^{max} = \{ (0, 0), \left( 0, \frac{1}{3} \right), \left( 0, \frac{2}{3} \right), \left( \frac{1}{3}, 0 \right), \left( \frac{1}{3}, \frac{1}{3} \right), \left( \frac{1}{3}, \frac{2}{3} \right), \left( \frac{2}{3}, 0 \right), \left( \frac{2}{3}, \frac{1}{3} \right), \left( \frac{2}{3}, \frac{2}{3} \right) \} \]

This group is the maximal symmetry group of \( W \), \( G^{max} \). It is not, however, the only possible group choice. The minimal symmetrical group is called \( J \) and is generated by the weights of \( W \). For the polynomials \( W = x^3 + y^3 \), the minimal symmetry group is generated by \( (\frac{1}{3}, \frac{1}{3}) \). In general, we can choose any group \( G \) where

\[ J \leq G \leq G^{max} \]

One method to find \( G^{max} \), is to first create a matrix \( A \) of the weights of the exponents of \( W \). The rows of \( A \) represent the monomials while the columns of \( A \) represent the weights of the exponents. For example, if we were looking at the polynomial

\[ W = x^3 y + y^2 z + z^5 \]

then the exponent matrix would be

\[
\begin{bmatrix}
3 & 1 & 0 \\
0 & 2 & 1 \\
0 & 0 & 5 \\
\end{bmatrix}
\]

The maximal symmetry group, \( G^{max} \), is then generated by the columns of \( A^{-1} \).

Once we have calculated the group, we are now ready to construct the building blocks of the vector space: Milnor rings. To calculate the Milnor rings of a polynomial \( W \) we use the formula

\[ Q_W = \left( \frac{C[x_i]dx_i}{\langle \partial W/\partial x_i \rangle} \right) \]
In this equation \( x_i \) are the variables appearing in the polynomial \( W \) and \( \langle \partial W / \partial x_i \rangle \) is the ideal generated by the partial derivatives of \( W \). We then compute the Milnor ring associated with each group element. If \( W_g = W \) restricted to only variables “fixed” by \( g \), then the direct sum of the Milnor rings is

\[
A_{W,G} = \bigoplus_{g \in G} \left( \frac{\mathbb{C}[x_i]dx_i}{\langle \partial W / \partial x_i \rangle} \right)^G
\]  

(1)

where

\[
R^G = \{ \text{elements of } R \text{ fixed by the action of } G \}
\]

We will now illustrate how this is done on the polynomial \( W = x^3 + y^3 \) with the group \( G^{\max} \). We will treat the group elements in two cases: the broad elements which contain zero entries, and the narrow elements which do not. When we plug these into the formula for the Milnor ring, we find a vector space of dimension four. This will be further explained in the next section when we discuss Fermat polynomials specifically.

As the name mirror symmetry implies, there is a dual model to this A model. The B model uses the dual of \( W \) and \( G \) to create another vector space. This has already been proven to be the same vector space as the A model. Therefore, this thesis will focus on computations in the A model as it is already known that as vector spaces they are equal to the B model. The A and B models can also be endowed with products and other complex structures. The duality of the A and B models is only partially proven in these cases.

2.2 New computations and proofs

The following theorem will be used in the subsequent proofs.

**Theorem 2.3.** ([4]) The additive basis for the Milnor ring of a Fermat polynomial of form \( x^a \), is given by elements of the form \( x^b dx \), where \( b \leq a - 2 \).
Theorem 2.4. The dimension of the Landau-Ginzburg vector space, $A_{W,G}$, formed by the polynomial $W = x^n + y^n$ with group $G = G^{\text{max}} = \langle (0, \frac{1}{n}), (\frac{1}{n}, 0) \rangle$ is $(n - 1)^2$.

Proof. Let $w = x^n + y^n$ where $n \in \mathbb{Z}$, $n \geq 2$, and $G^{\text{max}} = \langle (0, \frac{1}{n}), (\frac{1}{n}, 0) \rangle$.

Case 1: First we will look at the element $g = (0, 0)$. In this case, $W_g = W$ and both $x$ and $y$ are fixed by $g$. We then calculate the Milnor ring

$$Q_W = \left( \frac{\mathbb{C}[x, y] dxdy}{(nx^{n-1}, nyn^{n-1})} \right).$$

To see if this contributes to the basis, we will use Theorem 2.3. The basis elements $m$ in $Q_W$ will look like $m = x_1^{a_1} x_2^{a_2} dx_1 dx_2$ with $a_1, a_2 \leq n - 2$. In Equation (1) we see that in order for any of these basis monomials to appear in $A_{W,G}$, $g \times m$ must equal $m$ for any $g \in G$. This means that $(e^{\frac{2\pi i}{n} x})^{a_1} (e^{\frac{2\pi i}{n} y})^{a_2} (e^{\frac{2\pi i}{n} dx}) = x^{a_1} x^{a_2} dx dy$.

We can simplify this to

$$(e^{\frac{2\pi i}{n} a_1 + 1}) x_1^{a_1} \partial x_1 (e^{\frac{2\pi i}{n} a_2 + 1}) x_2^{a_2} \partial x_2 = x_1^{a_1} \partial x_1 x_2^{a_2} \partial x_2.$$

This can only happen if

$$\frac{a_1 + 1}{n}, \frac{a_2 + 1}{n} \in \mathbb{Z}$$

but because $a_1, a_2 \leq n - 2$ this is not possible. Therefore, this will not contribute to the basis.

Case 2: The second case is when $g = (\frac{a}{n}, \frac{b}{n})$, where $b, a \in \{1, \ldots n - 1\}$. In this case, $W_g = 0$ since there are no variables that are fixed by $g$. The formula for the Milnor ring then will give

$$Q_W = \left( \frac{\mathbb{C}}{\langle 0 \rangle} \right)$$

as there are no variables that are fixed. We will then obtain 1 copy of the complex numbers for each group element. The number of elements that have this form is $(n - 1)(n - 1) = (n - 1)^2$, this case contributes $(n - 1)^2$ elements to the basis.
2.2 New computations and proofs

Case 3: The third case is if \( g = \left( \frac{a}{k}, 0 \right) \) where \( a \in \{1, \ldots, k-1\} \). Then \( W_g = y^n \), and \( y \) is the only variable that is fixed.

We then calculate the Milnor ring for this case as

\[
\mathcal{Q}_W = \left( \frac{\mathbb{C}[y] dy}{\langle ny^{n-1} \rangle} \right)^G.
\]

To see if this contributes to the basis, we will use Theorem 2.3. All elements \( m \) in \( \mathcal{Q}_W \) will look like \( m = y^{a_2} dy \) for \( a_2 \leq k - 2 \). In order for this to contribute to the basis, \( g \times m \) must equal \( m \) for all \( g \in G \). This means that \( (e^{\frac{2\pi}{n} i} y)^{a_2} \left( e^{\frac{2\pi}{n} i} dy \right) = ya_2 dy \). We can simplify this to

\[
\left( e^{2\pi i \frac{a_2+1}{n}} \right) y^{a_2} dy = y^{a_2} dy.
\]

This can only happen if

\[
\frac{a_2+1}{k} \in \mathbb{Z},
\]

but because \( a_2 \leq n - 2 \) this is not possible. Therefore, this will not contribute to the basis.

Case 4: Finally, our fourth case will be if \( g = (0, \frac{b}{n}) \) where \( b \neq 0, n \). Then \( W_g = x^n \), and \( x \) is the only variable that is fixed. Using the exact same argument in Case 3, this will not contribute to the basis of \( A_{W,G} \).

Therefore, the dimension of the vector space of a Fermat polynomial with repeated powers is the direct sum of the Milnor rings which gives the formula \( \dim(A_{w,g}) = (n - 1)^2 \). We can see in this proof that whenever the group element had a zero, that group element did not contribute to the basis of the vector space.

Next we will give a generalization of the previous result.

**Theorem 2.5.** The dimension of the vector space formed by the polynomial \( W = x_1^k + x_2^k + \ldots x_n^k \) with \( G^{\text{max}} = \langle (\frac{1}{k}, 0, 0, \ldots, 0), (0, 0, \ldots, 1, 0), (0, 0, 0, \ldots, \frac{1}{k}) \rangle \) is given by the formula \( (k - 1)^n \).
Proof. From the above proof, we can see that if we are using a Fermat polynomial, then the broad elements (elements that contain a zero) will not fix any elements. This is because all elements $m$ in $Q_W$ will look like $m = x_1^{a_1}, ..., x_n^{a_n}dx_1...dx_n$. In order for this to contribute to the basis of $A_{W,G}$, $g \times m = m$ for all $g \in G$. This means that 

$$
\left( e^{2\pi i/k} \right)^{a_i} \left( e^{2\pi i} dx_i \right) = x_i^{a_i}dx_i
$$

as the other elements are not rotated. We can simplify this to

$$
\left( e^{2\pi i a_1/k} \right) x_i^{a_i}dx_i = x_i^{a_i}dx_i.
$$

This can only happen if

$$
\frac{a_1 + 1}{k} \in \mathbb{Z},
$$

but because $a_i \leq k - 2$ this is not possible. Therefore, this will not contribute to the basis.

We will next look at the narrow element, which will have the form $g = (\frac{a_1}{n}, \frac{a_2}{n}, ..., \frac{a_k}{n})$. These will never fix any variables as no entry is equal to zero. Therefore, $W_g = 0$. This means our ring formed will be

$$
Q_w = \left( \frac{C}{\langle 0 \rangle} \right)^G.
$$

and will contribute one copy of the complex numbers for each narrow group element.

Now we must solve for the number of narrow elements to see how large our basis will be. That is, if $g = (\frac{a_1}{k}, \frac{a_2}{k}, ..., \frac{a_k}{k})$ with each $a_i \in \{1, ..., k - 1\}$ then there are $(k - 1)^n$ unique such $g$’s by the multiplication principle of counting.

The subsequent theorem will be used in the following two proofs.

**Theorem 2.6.** ([4]) The Milnor ring of a loop polynomial $W$ of form

$$
W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + ... + x_k^{a_k}x_1
$$

is...
2.2 New computations and proofs

Romano has an additive basis made up of all elements

\[ Q_W = \langle x_1^{b_1}x_2^{b_2}x_3^{b_3}...x_k^{b_k}dx_1...dx_n \rangle \]

where \( b_i \leq a_i - 1 \).

**Theorem 2.7.** The dimension of the vector space of a loop polynomial of the form \( W = x^n y + y^n x \) under \( J = \langle \frac{1}{n+1}, \frac{1}{n+1} \rangle \) will be \( 2n \).

**Proof.** In order to prove this, we will see what happens to the broad and narrow elements of the group \( J \).

**Case 1:** First we will look at the broad elements. The element \( g = (0,0) \) is the only one which contains any zero entries.

For the broad elements, both \( x \) and \( y \) are fixed variables. This means that \( W_g = W \). We then calculate our Milnor ring as

\[ Q_W = \left( \mathbb{C}[x,y]dx dy \right) \left( \frac{nx^{n-1}y + y^n, x^n + xny^{n-1}}{n} \right). \]

We can then use Theorem 2.6 to note that the basis elements of the Milnor ring, \( m \), are the form

\[ m = x_1^{b_1}x_2^{b_2}dx_1dx_2. \]

where \( b_1 \leq a_1 - 1 \) and \( b_2 \leq a_2 - 1 \). Recall that we only use the parts of \( m \) which are fixed by \( G \). If \( g = \left( \frac{1}{n+1}, \frac{1}{n+1} \right) \), then \( g \times m \) will simplify to

\[ e^{2\pi i \left( \frac{b_1}{n+1} + \frac{b_2}{n+1} + \frac{x}{n+1} \right)} \]

We must now find the values of \( b_1 \) and \( b_2 \) which will fix this.

In order to fix this, we need to have \( \frac{b_1 + b_2 + 2}{n+1} \) equal 1. This is equivalent to saying \( b_1 + b_2 = n - 1 \). Using Theorem 2.7 we see that there are \( n \) solution to this as there
are \( n \) options for \( b_1 \) which then determines \( b_2 \). Therefore, the broad elements will contribute \( n \) elements.

For the narrow elements, no variables are fixed so \( W_g = 0 \). There will be \( n \) narrow elements because there are a total of \( n + 1 \) elements minus the one broad element. The Milnor ring can then be calculated as

\[
Q_W = \left( \frac{\mathbb{C}}{(0)} \right).
\]

This shows that every narrow element will contribute one copy of the complex numbers. Because there are \( n \) independent narrow elements, there will be \( n \) elements added to the basis. The dimension of the vector space \( A_{W,G} \) is the sum of the Milnor rings [7], so the dimension of a loop polynomial of the form \( W = x^n y + y^n x \) under \( J = \langle \frac{1}{n+1}, \frac{1}{n+1} \rangle \) will be \( 2n \).

\[\square\]

**Theorem 2.8.** The dimension of the vector space \( A_{W,G} \) of a loop polynomial of the form \( W = x^n y + y^n x \) under \( G^{\text{max}} = \langle \frac{-1}{n^2-1}, \frac{n}{n^2-1} \rangle \) will be \( n^2 \).

**Proof.** First we will prove that \( G^{\text{max}} = \langle \frac{-1}{n^2-1}, \frac{n}{n^2-1} \rangle \), and we will solve for its dimension. To solve for \( G^{\text{max}} \), we must first create the exponent matrix for our polynomial. This is

\[
\begin{bmatrix}
n & 1 \\
1 & n
\end{bmatrix}
\]

We then must invert the matrix which gives us

\[
\begin{bmatrix}
n & -1 \\
\frac{-1}{n^2-1} & \frac{n}{n^2-1}
\end{bmatrix}
\]

From this we can find the generators of the group which are the columns of the inverted matrix. Because the first and second columns are not linearly independent,
the generator of the maximal symmetry group is \(\langle \frac{-1}{n^2-1}, \frac{n}{n^2-1} \rangle\). Therefore, because there is only one generator the group \(G^{\text{max}}\) is a cyclical group, and the order is \(n^2 - 1\).

Further we know that there is only one point when the group is equal to \((0, 0)\) because \(n\) and \(n^2 - 1\) are relatively prime.

We can then split our elements into broad and narrow elements.

**Case 1:** \(g = (0, 0)\) There is only one broad element which is \((0, 0)\). For the broad element, both \(x\) and \(y\) are fixed variables. This means that \(W_g = W\). The Milnor ring then becomes

\[
Q_W = \left( \frac{\mathbb{C}[x, y]dx dy}{\langle nx^{n-1}y + y^n, x^n + xny^{n-1} \rangle} \right).
\]

We also know from Theorem 2.7 that

\[
Q_W = \langle x_1^{b_1}x_2^{b_2}dx_1dx_2 \rangle
\]

where \(b_i \leq a_i - 1\) [7]. We must now find the values of \(b_1\) and \(b_2\) in the following expression which fix this.

\[
e^{2\pi i \left(\frac{-b_1}{n^2-1} + \frac{nb_2}{n^2-1} + \frac{n-1}{n^2-1}\right)}
\]

This means that

\[
nb_2 - b_1 + n - 1 = 0 \mod (n^2 - 1).
\]

If we set this equation equal to zero, then this simplifies to

\[
n(b_2 + 1) = b_1 + 1.
\]

This can occur if \(b_1 = n - 1\) and \(b_2 = 0\) or if \(b_2 = n - 1\) and \(b_1 = 0\). This can also be fixed if

\[
nb_2 - b_1 + n - 1 = n^2 - 1
\]
but, we will show is not possible. This would mean that

\[ nb_2 + n - b_1 = n^2 \]

which results again in \( b_1 = 0 \) and \( b_2 = n - 1 \). Because this is fixed by 2 elements, we then know that there are 2 broad elements that are added to the basis.

**Case 2:** \( g = \left( \frac{a}{n^2-1}, \frac{b}{n^2-1} \right) \) For the narrow elements, neither \( x \) or \( y \) are fixed which means that \( W_g = 0 \). We then calculate our Milnor ring as

\[ Q_w = \left( \frac{\mathbb{C}}{\langle 0 \rangle} \right). \]

Therefore, this will contribute one copy of the complex numbers for each narrow element in the group. There are \( n^2 - 2 \) narrow elements, as the group has dimension \( n^2 - 1 \) and there is one broad element. Therefore, the narrow elements will contribute \( n^2 - 2 \) elements.

Therefore, \( \dim(A_{W,G}) = n^2 - 2 + 2 = n^2 \). \( \square \)

On interesting question that arises from the study of the dimension of these vector spaces is how many different types of each vector space there are. For most vector spaces, if the number of basis elements is the same, then they are isomorphic. However, Landau-Ginzburg vector spaces are also graded vector spaces, which means that the basis elements in these spaces all have a degree associated with them which distinguishes the vector space beyond the size of the basis. On familiar example of a graded vector space is \( \mathbb{R}[x] \) which is graded in the degree of the polynomial.

The question we wanted to answer was how often does a four dimensional vector space arise and are they all the same space. Initially, I conjectured that there was only one type of four dimensional vector space, and that it only arose a few times. However, I have calculated all the vector spaces for two and three variable Fermat, loop, and chain polynomials with exponents up to eight, and have found 23 times
that four dimensional vector spaces arises.

These 23 spaces are not all the same graded vector space either. They occur in four distinct categories based on the degree of each basis element. Table 1 shows each polynomial which gives a four dimensional vector space, the group used, the degrees of each element, and the number of broad elements in the space. If we look at the space with degrees \((0, \frac{2}{3}, \frac{4}{3})\) it is interesting that the space can have zero, one, or two broad elements in its basis. However when we look at the space with degrees \((0, 1, 1, 2)\) there are always two broad elements and in the space with degrees \((0, 0, \frac{2}{3}; \frac{2}{3})\) and \((0, \frac{2}{5}; \frac{4}{5}; \frac{6}{5})\), there are always zero broad elements.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Group</th>
<th>Degrees</th>
<th>Number of broad elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^5)</td>
<td>(\langle 1/5 \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>0</td>
</tr>
<tr>
<td>(x^5 + y^2)</td>
<td>(\langle 1/6, 1/2 \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>1</td>
</tr>
<tr>
<td>(x^2y + xy^2)</td>
<td>(\langle 1/3, 1/3 \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>2</td>
</tr>
<tr>
<td>(x^3 + y^3)</td>
<td>(\langle 1/3, 1/3 \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>2</td>
</tr>
<tr>
<td>(x^2 + y^3)</td>
<td>(\langle (0, 1/3), (1/3, 0) \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>0</td>
</tr>
<tr>
<td>(x^2 + y^4)</td>
<td>(\langle 1/3, 1/3 \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>2</td>
</tr>
<tr>
<td>(x^2 + y + y^2)</td>
<td>(\langle 1/6, 1/2 \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>0</td>
</tr>
<tr>
<td>(x^5 + y^2)</td>
<td>(\langle 1/2, 3/5 \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>0</td>
</tr>
<tr>
<td>(x^3 + y^2 + z^5)</td>
<td>(\langle 3/5, 1/2 \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>0</td>
</tr>
<tr>
<td>(x^3 + y^5 + z^3)</td>
<td>(\langle (0, 0, 1/3), (1/2, 2, 3/0) \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>0</td>
</tr>
<tr>
<td>(x^3 + y^2 + z^3)</td>
<td>(\langle (0, 0, 1/3), (2/3, 1/2, 0) \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>0</td>
</tr>
<tr>
<td>(x^3 + y^2 + z^2)</td>
<td>(\langle (1/2, 0, 0), (0, 3/5, 1/2) \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>0</td>
</tr>
<tr>
<td>(x^4 + y^2 + z^2)</td>
<td>(\langle (0, 1/3, 0), (2/3, 0, 1/2) \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>0</td>
</tr>
<tr>
<td>(x^3 + y^2 + z)</td>
<td>(\langle (0, 1/2, 0), (3/5, 0, 1/2) \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>0</td>
</tr>
<tr>
<td>(x^4 + y^2 + z^4)</td>
<td>(\langle 1/4, 1/2, 1/4 \rangle)</td>
<td>((0, 1, 1, 2))</td>
<td>2</td>
</tr>
<tr>
<td>(x^3 + y^2 + z^3)</td>
<td>(\langle 1/3, 1/2, 1/3 \rangle)</td>
<td>((0, 1, 1, 2))</td>
<td>2</td>
</tr>
<tr>
<td>(x^4 + y^2 + z^3)</td>
<td>(\langle 1/2, 1/2, 2/3 \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>0</td>
</tr>
<tr>
<td>(x^4 + y^2 + z^2)</td>
<td>(\langle 2/3, 2/3, 1/2 \rangle)</td>
<td>((0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}))</td>
<td>0</td>
</tr>
<tr>
<td>(x^3 + y^2 + z^2)</td>
<td>(\langle 1/3, 1/3, 1/3 \rangle)</td>
<td>((0, 1, 1, 2))</td>
<td>2</td>
</tr>
<tr>
<td>(x^2 + y^2 + z^2)</td>
<td>(\langle 1/3, 1/3, 1/3 \rangle)</td>
<td>((0, 1, 1, 2))</td>
<td>2</td>
</tr>
<tr>
<td>(x^4 y + y^2 + z^2)</td>
<td>(\langle 1/4, 1/4, 1/2 \rangle)</td>
<td>((0, 1, 1, 2))</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1: All two and three variable Fermat, loop, and chain polynomial with exponents up to eight yielding four dimensional vector spaces.

This observation led to a simplification in the problem. If we restrict the poly-
mial used to a two variable Fermat polynomial, we can obtain the following theorem.

**Theorem 2.9.** There exist no dimension four vector spaces constructed by Fermat polynomials with two variables and an exponent greater than eight.

**Proof.** In order to prove this we will consider two cases and how many narrow elements are in each vector space. Each narrow element of a Fermat polynomial always contributes to the vector space as shown in previous proofs. Therefore, if we can show that the number of narrow elements exceeds four, then we have completed the proof.

Without loss of generality, let $2 \leq a \leq b$. We will now solve for the value of $b$ which will guarantee that we have a vector space of dimension greater than four. The group $J$ will be $\langle \frac{1}{a}, \frac{1}{b} \rangle$ and will have dimension of $\frac{ab}{\gcd(a,b)}$. We are testing the dimension of $J$ because $J$ is always contained in any symmetry group we can use. Therefore if the vector space created by only the narrow elements $J$ is greater than four, then any vector space using this polynomial and any group will have dimension greater than four.

Next, to simplify the problem, we will calculate only the number of narrow elements that will be in our basis. If we can prove that the number of narrow elements that contribute to the basis of $A_{W,G}$ is greater than four, then we have completed the proof. The number of narrow elements will be equal to the total dimension of $J$ minus the number of broad elements. The number of broad elements will be

$$a \frac{1}{\gcd(a,b)} + b \frac{1}{\gcd(a,b)} - 1.$$

We need to subtract one for the element that is $ab$ as this would otherwise be over counted. Therefore, the number of narrow elements will be

$$\frac{ab}{\gcd(a,b)} - \left( a \frac{1}{\gcd(a,b)} + b \frac{1}{\gcd(a,b)} - 1 \right).$$
First we will rewrite the number of narrow elements as

\[ \text{lcm}(a, b) - \frac{a}{\text{gcd}(a, b)} - \frac{b}{\text{gcf}(a, b)} + 1, \]

which we can do because of the relationship between least common multiples and greatest common divisors. Next we can say that \( \text{lcm}(a, b) \geq \text{max}(a, b) \) and that \( \frac{b}{\text{gcf}(a, b)} \geq \frac{b}{\text{min}(a, b)} \) and similarly for the \( a \) term. Therefore,

\[ \text{lcm}(a, b) - \frac{a}{\text{gcd}(a, b)} - \frac{b}{\text{gcf}(a, b)} + 1 \geq \text{max}(a, b) - \frac{b}{\text{min}(a, b)} - \frac{a}{\text{min}(a, b)} + 1 \]

Based on how we defined \( a \) and \( b \), we know that the minimum will be \( a \) and the maximum will be \( b \), so we can then simplify our statement to

\[ b - \frac{b}{a} - \frac{a}{a} + 1 \]

which will then simplify to

\[ b - \frac{b}{a} = b \left( 1 - \frac{1}{a} \right) \]

Because the smallest \( a \) can be is two, we must solve for the value of \( b \) which will guarantee that we have more than four narrow elements. That is

\[ 4 \geq b \left( 1 - \frac{1}{2} \right). \]

Therefore, if we let \( b > 8 \), we guarantee that there are more than four narrow element and the vector space will have a dimension larger than four.

\[ \square \]

**Theorem 2.10.** Solving for the \( A \) model of a Fermat polynomial with matching exponents \( a \) under \( J \) is equivalent to solving for the \( A \) model of a loop polynomial with matching exponents of \( a - 1 \) under \( J \).
Proof. The proof of this will consist of making the Milnor rings for a loop polynomial with matching exponents of $a - 1$ with $J$ and the A model of a Fermat polynomial with matching exponents of $a$ with $J$ and showing they are equivalent.

First for the loop, $x_1^{a-1}x_2 + x_2^{a-1}x_3 + ... + x_n^{a-1}x_1$, the group $J$ will be the cyclic group given by

$$ \left\langle \frac{1}{(a-1)+1}, \frac{1}{(a-1)+1}, \ldots, \frac{1}{(a-1)+1} \right\rangle. $$

For the Fermat polynomial, $x_1^a + x_2^a + x_3^a + ... + x_n^a$, the group $J$ will also be a cyclic group given by

$$ \left\langle \frac{1}{a}, \frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a} \right\rangle. $$

The only element in either group that can fix anything will be the broad element $g = (0, 0, 0, ..., 0)$ which is where $W_g = W$ as shown in prior proofs.

Then by Theorems 2.3 and 2.7, $Q_W = \langle x_1^{b_1}, x_2^{b_2}, x_3^{b_3}, ..., x_n^{b_n} \rangle$ where $b \leq (a - 1) - 1$ for the loop, and where $b \leq (a - 2)$ for the Fermat. Therefore the rest of the A model for these two are the exact same problem as the Milnor rings are the same. This proves that solving a loop polynomial with matching exponents of $a - 1$ under the group $J$ is equivalent to solving a Fermat polynomial with matching exponents of $a$ under $J$.

\[\square\]

3 Future work

Starting with the definitions from Landau-Ginzburg mirror symmetry we were able to prove new results about the dimensionality of vector spaces that are generated. We were able to prove the dimension of vector spaces for Fermat polynomials with group $G^{max}$, two variable loops with groups $J$ and $G^{max}$, and finally some conclusions on when 4 dimensional vector spaces arise. Future work in this field could include a
more generalized proof of 4 dimensional vector spaces as we were only able to solve for two variable Fermats. It could also include proving the dimension of n variable loop polynomials using both $J$ and $G^{max}$. 
References


