Suggested Types of Manipulatives to Aid in Teaching Geometry to Various Age Groups of Students

Tessa Berg
Carroll College 2015

Director:
Dr. Joe Helbling

Readers:
Dr. Kelly Cline
Dr. Eric Sullivan

SIGNATURE PAGE

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Director Dr. Joe Helbling — **Date**

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Reader Dr. Eric Sullivan — **Date**

3/19/2015

Reader Dr. Kelly Cline — **Date**

ABSTRACT

Geometry is a complex and visual discipline. Because geometry includes abstract concepts, geometry teachers often use manipulatives to make concepts visible and tangible. In this thesis, the question is proposed: which types of manipulatives are most beneficial to which age groups based on student achievement, student engagement and the Common Core State Standards. This thesis analyzes the existing research about the use of different types of manipulatives with different age groups. The research was separated into age group brackets, kindergarten through second grade, third grade through fifth grade, sixth grade through eighth grade, and ninth grade through twelfth grade. The types of manipulatives were categorized into one of four types of manipulatives: Physical Designated (use determined by teacher), Physical Non-Designated (use left for the student to decide), Virtual Designated, and Virtual Non-Designated. For the age group including students in kindergarten through second grade, Physical Designated and Virtual Designated manipulatives were suggested based on research. For the age group including students in third through fifth grade, Physical Designated, Physical Non-Designated, and Virtual Designated manipulatives were suggested. For the age group sixth through eighth grade all four types of manipulatives were suggested. For the ninth through twelfth grade age group, Physical Non-Designated, Virtual Designated, and Virtual Non-Designated manipulatives were suggested.

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Chapter 1: Introduction

There are four major sections of this introduction chapter. The first section provides a background and purpose for the question addressed by the research. The second and third section discusses the context of the question and practical outcomes of the study. The final section outlines the significance and scope of the research question. This introduction chapter provides a framework for the research question.

Background and Purpose

Geometry exists all around us. Consequently, educators of mathematics are encouraged and required to teach geometry at nearly every age. In 2010, the Common Core State Standards were developed in mathematics (including geometry standards) for students in grades kindergarten through 12th grade (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).   There are 83 Common Core State Standards related to geometry that students are expected to learn between kindergarten and 12th grade. There is, however, no common means of teaching these standards. As a result, many educators develop their own strategies to teach the complexities of geometry.

Having investigated, implemented, and evaluated numerous ways to teach geometry, many modern educators heavily rely on manipulatives. Geometry is an visual and physical discipline and many educators believe it is best taught through visual and kinesthetic lessons. From geometry tiles, to geoboards, to computer-based geometry programs, geometry manipulatives have been studied extensively. Based on its widespread use, there is a great need for educators of all age levels to better understand and utilize the resource of geometry manipulatives. Unfortunately, with so many resources available to educators, it can be difficult to determine where to begin.

Educators may have numerous questions regarding manipulatives. Which types of manipulatives should I use for my students at my grade level? Which types of manipulatives are best aligned to the Common Core Curriculum? Which types of manipulatives are most effective for specific standards? Are some types of manipulatives inappropriate for various age levels? The confusion stems from an issue with current research about geometry manipulatives. Most research studies investigate the success of a specific manipulative with a specific age group. A few studies may even compare the success of two manipulatives. No research study can simply answer the question most educators have, which manipulatives should I use for *my* specific students? To answer this, the research question is proposed and defined in the context section.

Context

Based on the background of manipulatives in the geometry classroom, the following research question is proposed: What specific types of manipulatives are most appropriate to different grade levels according to 1) their benefit to student learning, and 2) student engagement, and 3) alignment to Common Core Standards. The research question will guide the synthesis of major studies to provide educators with a more clear answer to the question of which manipulatives to use within their classroom. First, this study will investigate the need for manipulatives in the geometry classroom based on studies of cognitive development and studies regarding the success of manipulatives in improving student achievement. Next, the studies regarding the use of specific types of manipulatives will be analyzed for how the manipulative affects student engagement and motivation. Finally, the types manipulatives found to improve student achievement and student engagement will be compared to the Common Core State Standards to determine which types of manipulatives are most applicable to meet the learning standards.

Significance and Scope

To begin research on the question, one must first understand the significance of the question. To answer the research question of which types of geometry manipulatives are most appropriate for specific grade levels based on the three criteria of standards alignment, student achievement, and student engagement, means to provide a guide for geometry educators to choose from the hundreds of geometry manipulatives available today. While the teacher must make the final decision about which manipulatives to include in their classroom, the research question will help to provide a useful starting point.

The research done is distinct from other research in the field of geometry manipulatives because it accounts for not only student success but also what students are required to know and how students are motivated to learn. Rather than rating the success of a specific manipulative, this study suggests various types manipulatives that will align with improving student achievement, motivation and meeting the Common Core Standards as resources for teaching geometry to specific grade levels.

The scope of the study is not to prove one manipulative better than another, but rather to provide educators with a list of types of manipulatives which they should feel confident utilizing in their grade level classroom. The scope of the research question is narrowed by type of manipulatives (physical designated, physical non-designated, virtual designated or virtual nondesignated [for a definition of each type of manipulative, see appendix I]) and grade level groups. Further narrowing of the scope is outlined in Chapter 4 of the thesis.

Chapter 2: Literature Review

In this literature review chapter, the previous research and discussion of the use of geometry manipulatives in the classroom will be synthesized and evaluated. First, the historical background of the research on the use of geometry manipulatives will be briefly summarized. Then, the gradual change of the research over time will be addressed. In this part of the chapter, the change in the research over time will be demonstrated as parallel to the changing needs of the modern mathematics educator. Next, the importance of the research will be explained in the context of the issues in teaching geometry to various groups of students. There are four major issues educators face when teaching mathematics to their students: students' cognitive ability to visualize complex problems, students' motivation, aligning educational techniques to Common Core standards, and teaching to different groups of students with different learning needs. The research addressing each of these issues will be investigated, compiled, and analyzed. A summary of all the previous research and its implications will conclude the chapter.

Manipulatives have been used to teach geometry for many generations. However, the study of the use of manipulatives has only been developing since the late 1970s (Mooyer, 1978). The early research done on the use of manipulatives in the classroom was very broad and served the primary purpose of justifying hands-on learning based solely on student achievement (Sowell, 1989). Early research seems to suggest the primary reason for studying the use of manipulatives in the geometry classroom was based on a common belief among mathematics educators that various groups of students lacked the cognitive ability to conceptualize complex geometry problems (Mooyer, 1978 and Sowell, 1989). Geometry, as defined by Merriam Webster Dictionary (n.d.), is a branch of mathematics that deals with the measurement, properties, and relationships of points, lines, angles, surfaces, and solids. Therefore, geometry, by definition, is a visual and physical discipline.

Before educators realized geometry was a complex discipline for a child to conceptualize, Jean Piaget had made groundbreaking discoveries about the cognitive development of a child. In his most famous work, The Psychology of the Child , Piaget studied why children gave different answers than adults to questions requiring logical thinking (1969). Prior to Piaget's study, many psychologists believed that children were simply less intelligent and less abled thinkers than were adults. Piaget's work showed that children were no less competent in their ability to think through a problem, they merely looked at the problem in a different manner. Piaget's work is fundamental for geometry educators. As adults, educators believe there are a few logical paths to solving a problem. A child may think of a geometry problem in a manner an educator never thought possible.

Piaget also explains that children are born with basic cognitive abilities. The more complex abilities are learned and developed throughout childhood and into adulthood. In particular, from age 2 to 7 years, most children are in the preoperational stage of development. During this stage, children can begin using concrete and physical manipulatives to understand complex problem solving. A child in this stage can understand the use of objects to represent other objects during play. However, most of the child's thinking is centered around themselves so the symbolism is related to something familiar. A child in this stage is likely to believe that when an object changes form, it is no longer the same object. Consequently, complex thinking at this stage can be very difficult because the child will easily lose track of what each object represents.

From age 7 to 11 years, the majority of children are in the concrete operational stage of development. During this stage of development, children have a better understanding of concrete examples and symbolism of objects. Still, the use of concrete and physical manipulatives is the primary way children learn. At this stage, children are not able to visualize complex problems in their head. After the concrete operational stage, children enter the formal operational stage (usually around age 11 or 12). In the formal operational stage, children can think in a more abstract context. Children in this stage are able to visualize a problem without manipulatives. Nevertheless, a child in this stage will still need to use visual manipulatives in their mind to reason the answer to complex problems. As a result, for the child to reach this stage of development, it is imperative they are taught how to utilize manipulatives during the concrete operational stage of development.

Piaget's work established a foundation for geometry educators. Children must be taught the use of manipulatives throughout their development in order for them to problem-solve without manipulatives (mental problem solving). Piaget's explanation of childhood development parallels the findings of geometry educators today. Areti Panaoura studied the cognitive development of a children's geometric thinking through manipultives and found that conceptualizing geometry without the use of visuals and/or physical manipulatives is beyond the cognitive ability of students under the age of 14. Some students will not gain the cognition to visualize complex geometrical problems until they reach adulthood (2014). While studies like Panaoura's and Piaget's suggest a lack of cognitive ability for complex visualization is normal, it does not mean students in the K-12 system have to miss out on learning complex geometrical concepts.

In fact, Lev Vygotsky, a famous child psychologist, writes that children are capable of very complex thinking if provided the opportunity to learn such types of thinking. In his Theory of Social Development, Vygotsky argued that children could learn complex thinking at any age if given the proper learning tools. Unlike Piaget, Vygotsky believed that learning and cognitive development was not related to age. According to Vygotsky, a child must learn and be taught how to think abstractly before developing higher-level thinking. The Theory of Social Development does not limit a child's ability based on age, but argues that a child's thinking is only as advanced as what the child is taught. Most importantly, the theory explains that a child learns from the culture around him/her. Therefore, if educators challenge a child with proper learning tools to develop his/her cognitive problem solving abilities, the child will be able to develop those abilities accordingly. For geometry educators, Vygotsky's theory shows the importance of introducing manipulatives to children throughout their development and not just when their age suggests it is appropriate. Children lack the cognitive ability to visualize geometrical problems throughout their development. However, educators can help children to develop that cognitive ability by providing opportunities for more complex learning.

For most educators, the remedy for the lack of cognitive ability to visualize geometrical problems is the use of hands-on manipulatives. Based on the studies in psychology by Piaget and Vygotsky as well as the studies conducted in the early research period of geometry manipulatives, educators are correct in acknowledging a need for manipulatives in teaching geometry. Several educational companies have developed hands-on manipulatives to help students visualize geometrical concepts. Many of these manipulatives are toy-like in nature but were developed with the intent of bringing complex geometry to the level of understanding of the average student. Early research argued that all students needed hands-on manipulatives to understand geometry, but as research developed, it was found that different manipulatives benefited different groups of students depending on age, ability, and interest (Perham, Perham, and Perham, 1997).

In fact, some research even discovered that certain groups of students did not benefit from the use of manipulatives. For example, in an article for School Science and Mathematics, Barbara Burns and Ellen Hamm developed an experiment to compare the effectiveness of virtual manipulatives in comparison to physical manipulatives on improving student learning in a sample of 91 third grade students and 54 fourth grade students. Students were split into three groups. The first group was not exposed to the use of any manipulatives, the second group was taught through the use of physical manipulatives, and the third group was taught through the use of virtual (computer based) manipulatives. The results of the study found there was no difference between the three groups in student achievement.

The findings of Burns and Hamm are different from the findings of Trisha Ainsa (1999). Ainsa conducted a study having students use either M&Ms as manipulatives or computer software in five different early childhood classrooms (ages 3-6). The students were to compare the size of various geometric shapes by modeling them with the software or with M&Ms. Ainsa's study found that students in both groups benefited from their respective manipulative (physical or virtual) because their performance on answering the same type of question was improved over time.

As a result of the wide variety of studies suggesting the need for manipulatives, such as Ainsa, and the variety of studies, such as Burns and Hamm, which were found inconclusive, many studies began to focus on the success of particular manipulatives. In 1989, Evelyn Sowell, Mathematics Professor at Governors State University, compiled 60 studies, which were completed to test the effectiveness of manipulative materials in learning geometry. The age of the subjects in the studies spanned from kindergarten to college. Each individual study found whether a particular manipulative was successful or not successful in improving student achievement. Findings about specific manipulatives were not always meaningful. As Sowell explained:

> During the 1960s and 1970s, researchers compared, in a number of educational settings, outcomes of mathematics instruction with concrete or pictorial materials to outcomes of instruction without such materials. Their results were often mixed. Findings in some comparisons favored the group using materials, whereas in other comparisons the control group achieved comparable or better results (pg. 498).

After Sowell's compilation of studies regarding specific manipulatives, even more studies were completed. These studies tested the success of various manipulatives for various groups of students at various skill levels. As these studies developed and research changed over time, educators found that specific manipulatives did improve student achievement. For example, Andreja Istenic Starcic, Mara Cotic, and Matej Zajc, associate professors of education at Ljubljana University in Russia conducted a study over a two-year period of 145 students (Starcic, Cotic, Zajc, 2013). The students ranged from low to high cognitive ability and low to high fine motor skills. The study was to determine student success in developing geometrical skills by combining cognitive thinking with physical manipulation. Forty-nine teachers were asked to implement specific physical and virtual manipulatives in their classroom along with traditional instruction. The teachers were asked to observe changes in their students' abilities, engagement and motivation. As the study was qualitative in nature, it could not numerically demonstrate any significant increase in student ability. However, a relationship was established between student success and the use of computer based and physical manipulatives. The forty-nine teachers were interviewed about their students success after the introduction and continued use of virtual and physical manipulatives. All forty-nine teachers agreed that the manipulatives improved not only their students understanding of the geometric concepts but also their self efficacy and engagement.

Overtime, research developed into studies similar to the work of Starcic, Cotic and Zajc. Research focused not only on improving student ability but also student motivation and engagement. Barbara Armstrong, Professor of Education at San Diego State University and Carol Larson, Professor of Education at the

University of Arizona, conducted a study of 12 fourth grade students, 12 sixth grade students and 12 eighth grade students. The students were asked to solve 21 comparison of area tasks during a clinical interview. In other words, the students were to compare the area of two or more shapes while being monitored by a test proctor. The areas of the shapes were partitioned into rectangles to assist the students in comparing the size of two shapes. Regardless of the partitioned rectangles provided, the students were allowed to solve/answer the problem using the strategy of their choice. The strategy each student utilized was placed into one of three main categories. The categories were as follows: Part-whole, Part-Whole/direct comparison, and Direct comparison. The part-whole category included responses in which students used the visual and/or representation of the shapes being divided into smaller parts to compare the areas. For example, the student's reasoning might be, the first shape fits four rectangles while the second shape fits 8 rectangles, thus the second shape must have a larger area. The direct comparison method included responses in which students used the visual and/or physical manipulation of literally comparing the size of two shapes. For example, the student's reasoning might be, when I place the first shape over the second shape, the first shape covers the entire second shape and then some, thus the first shape must have a larger area. The part-whole/direct comparison category was a combination of these two responses.

All 36 students who participated in the study were given the option of how to reason their response. All 36 students chose an approach, which included a visual and/or physical manipulative. Based on the grade level, students chose a strategy with which they felt they would be most successful in correctly answering the question. Armstrong and Larson noted the students in all three grade levels performed at grade level or above on comparing the area of the shapes. Student choice regarding which manipulatives and strategies to use also played a role in their success as well as their engagement in the lesson.

This study, along with others, marked a change in research regarding the use of manipulatives in teaching geometry. Student success was not the only variable researchers were interested in investigating student motivation and engagement also plays a role in the success of implementing geometry manipulatives in the classroom.

Research has changed over time since first showing that children cognitively benefit from physical manipulatives to develop higher-level thinking. This beginning research was supported by the work of Piaget and Vygotsky but affirmed by educators in the 1970s who found students who were able to learn through manipulatives tended to have a higher ability for geometric thinking than students who were not able to learn through manipulatives (Mooyer, 1978). These studies, demonstrating the benefit of implementing manipulatives in the classroom to improve student achievement continued through the 1980s (Sowell, 1989). During this time period, the main focus of the research regarded the use or lack of use of manipulatives. It wasn't until the late 1990s when research made another shift to focus on which manipulatives were more effective than others, such as the research of Burns and Hamm which compared the use of virtual and physical manipulatives. While their research proved inconclusive, it was one of the first to acknowledge the importance of distinguishing which manipulatives are, and which are not, beneficial to student achievement. Finally, research shifted to focus not only on student achievement but on student engagement. Research, such as the study conducted by Armstrong and Larson, suggested that educators should provide students with choice regarding which manipulatives and strategies to use. In the study by Armstrong and Larson, students seemed to perform well because they were more engaged in the question proposed to them. As research continues to change and develop, it tends to now focus on which manipulatives benefit student achievement and improve student engagement.

Still, for modern educators, the issues of student achievement and student engagement are not the only issues which need to be addressed when considering whether or not to utilize specific manipulatives in teaching geometry. In 2007, Arizona Governor Janet Napolitano, then chair of the National Governors Association, wrote an initiative for improving education, especially in science and math, across the country. The direction behind the initiative was to better prepare students for the workforce. This initiative proposed by Governor Napolitano became the foundation for the Common Core Standards (Bidwell, 2014). Every state in the United States has been encouraged to adopt the Common Core State Standards on the basis that education should be held to a higher standard throughout the country. For mathematics educators, there are specific standards regarding geometry for grades kindergarten through 8^{th} grade. There are also very detailed standards for geometry for students 9^{th} grade through 12^{th} grade which are not specifically tied to an age or a grade. At each grade level and the 9^{th} through 12^{th}

grade level grouping, there are standards which address the use of manipulatives and standards which do not specifically call for the use of manipulatives. Consequently, educators in the United States must determine which manipulatives are appropriate to fit these standards and when manipulatives need to be used for the standards, which are less explicit. Common Core State Standards now play a major role in what and how geometry educators teach and the decision of which manipulatives to use in the classroom must also be tied to the standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

In summary, there are three major issues addressed by research about geometry manipulatives in the classroom over the past several decades. The first issue addressed the need for manipulatives cognitively, the second issue addressed the success of using various types of manipulatives, and finally the third issue addressed student engagement and motivation. Alongside the three major shifts in the research, there was also an important development in the standards taught by educators across the country with the onset of the Common Core State Standards.

Chapter 3: Research and Methodology

In this chapter, the process of answering the research question will be outlined. First, the chapter will discuss the methodology and research approach. Next, the scope will be discussed. Finally, the limitations and weaknesses of the research will be addressed.

Methodology and Research Approach

To begin research in answering the question of which manipulatives are most appropriate for various age levels based on student achievement, student engagement and Common Core standards, the history of the research needed to be addressed as well as how research changed over time.

Prior to researching specific studies about the use of manipulatives to answer the research question, the research of a child's cognitive development was conducted. The early research of Piaget and Vygotsky in the field of child psychology was fundamental in the later research of educators regarding the need for manipulatives to develop a child's cognitive ability. In order to address the research question of which manipulatives are appropriate for teaching various age groups, it is crucial to first understand the cognitive needs of each age group. Both Piaget and Vygotsky argued that children needed manipulatives throughout their development. Piaget and Vygotsky differ in their opinion of how quickly manipulatives should be introduced throughout development. While Piaget believed that once a child reached a given age, they were able to use manipulatives to represent more complex problems, Vygotsky believed that a child's cognitive ability was not linked to age. In other words, Vygotsky believed that manipulatives should be introduced to solve complex problems as quickly as a child was able to master a problem of slightly lower complexity. Both the work of Piaget and Vygotsky demonstrate a need for manipulatives to improve learning and cognition throughout development. Both psychologists recognize the need for manipulatives to form higher level thinking amongst children.

With an understanding of the research of Piaget and Vygotsky, the research could move forward into investigating specific studies about the use of specific manipulatives. Based on the understanding of the child's cognitive development, the research was narrowed to investigate how specific types of manipulatives benefited students of specific age ranges. The research found educators have studied the result of using manipulatives to reach specific learning goals for students. Because geometry is a complex discipline, the use of manipulatives in teaching geometry has become a primary area of research for educators. When analyzing the research completed by these educators, three common issues were discovered in reference to the research question. First, the use of manipulatives and various types of manipulatives in teaching geometry is justified through studies based on student achievement (or improvement in student ability). Next, the use of various manipulatives is justified through studies based on student engagement. Finally, the use of various manipulatives is justified based on a need implied by the Common Core State Standards for Mathematics. As the research changes over time regarding geometry manipulatives, it parallels the important issues educators must consider when deciding which manipulatives to use in their classroom.

After developing a general understanding of the history and the change in the research over time, specific studies on specific manipulatives were analyzed. For example, the studies of Ainsa (1999) and Burns and Hamm (2011) discussed the use of physical manipulatives and virtual manipulatives. These studies were used to show how different manipulatives benefit the achievement of different groups of students. Still, other studies showed that allowing students choice in which manipulatives and problem solving strategies to use in reasoning a geometry problem improved student engagement. These studies were important to analyze because they demonstrated the need for various types of manipulatives to fit the learning styles of various types of students.

The majority of the research completed regarding the use of manipulatives in teaching geometry does not take into account the Common Core State Standards. The Common Core standards are an important issue for educators to consider when teaching mathematics. As explained in chapter 2, the Common Core standards were developed to improve education across the country. Educators must know the Common Core standards before implementing manipulatives in their classroom. Specific standards call for specific manipulatives while other standards could be taught through the use of manipulatives without explicitly stating so. The standards which could warrant the use of manipulatives must be outlined in order for educators to meet the standards and the needs of their students.

Therefore, the research was organized by first investigating the cognitive learning needs of students at specific ages. Then, the research of the use of specific types of manipulatives was compiled and organized by age group. The research was further broken down by type of manipulatives and whether or not the manipulative was beneficial to the learning needs of the students. In order to analyze these findings, both the specific studies regarding the use of the manipulatives as well as the general research about cognitive development were considered in suggesting specific types of manipulatives for specific age groups. Lastly, the types of manipulatives found to be suggested for specific age groups were compared to the Common Core Standards. If the type of manipulative met the learning needs outlined by the standards, than the type manipulative was suggested for that age group. If the type of manipulative did not meet the learning needs outlined by the standards, than the type of manipulative was not suggested for that age group.

Scope

To narrow the scope of the research question, grade brackets were created. The age brackets were necessary based on the theories of Piaget because children of different ages have different levels of cognitive ability. According to the theories of Vygotsky, the grade brackets established are based on an average student ability. Some students' need for specific types of manipulatives may fall above or below their suggested grade bracket group. The grade brackets established were loosely based on the developmental stages suggested by Piaget as well as the findings of research on the use of manipulatives in teaching geometry. The age brackets to be addressed are as follows: kindergarten through second grade, third grade through fifth grade, sixth grade through eighth grade, and ninth grade through twelfth grade. Having grades separated into brackets narrows the scope to focus on which manipulatives are specifically appropriate for which grades based on student achievement, student engagement, and Common Core standards.

Grouping specific manipulatives together by type will also narrow the scope. Because there are literally thousands of geometry manipulatives available for educators to use today, the research question must be narrowed to determining which types of manipulatives are most appropriate for each grade bracket. From the research, there are four primary types of geometry manipulatives used by educators today. The first type of manipulative are physical and designated manipulatives. These manipulatives are hands-on tools provided to students where the teacher guides how the manipulative is to be used to solve the problem. The second type of manipulative is physical and non-designated manipulatives. These manipulatives are also hands-on tools but the teacher does not designate how the manipulative is to be used to solve a problem. These manipulatives are used for student guided learning rather than teacher guided learning. The third type of manipulative are virtual and designated manipulatives. These manipulatives are still hands-on but are used through computer software. Much like the physical and designated manipulatives, the virtual and designated manipulatives are used when the teacher wants to instruct the student how to use the manipulative or the software program guides the student how to use the manipulative to solve specific problems. The fourth type of manipulative is virtual and non-designated. These manipulatives are used through computer software but are not guided by the teacher or the program as to how to solve problems. These types of manipulatives may also used for student guided learning.

Limitations

There are a few limitations in analyzing the research regarding the use of manipulatives to teach geometry. First, there is no common means of measuring student achievement. In some studies, improved student achievement is defined as a student's score on a specific test or problem improved after introducing the manipulative. In other studies, improved student achievement is defined as the student improving their performance on similar problems over an extended period of time. Still, in other studies, improved student achievement was defined as improved performance on several types of geometry problems after the use of manipulatives. While it is difficult to measure and define improved student achievement, the research can always be compared to the average performance of the student prior to the uses of the manipulative. In other words, a comparison must be made between the average grade level performance of the student and the performance of the specific student after the use of the manipulative.

Another potential limitation of the research is the possible sources of error in measuring the success of manipulatives. When a student is introduced to a manipulative, they are often given more guided instruction on how to solve a problem. It can be difficult to determine if the manipulative or the guided instruction improved the student's achievement. Furthermore, it is difficult to get a large enough sample in any study to determine statistical significance that one manipulative is more beneficial than another. The majority of studies have a relatively small sample size (less that 50 students) and contain a sample of similar students (of a similar background and cognitive ability). The studies about specific small populations cannot demonstrate with statistical significance that a manipulative will benefit all students of a similar age, but merely that the manipulative benefited that particular group. The studies can merely serve as suggestions to which manipulatives may benefit different age groups. These limitations to the studies conducted by educators suggest the need for a base understanding of the cognitive development of a child through the research of psychologists such as Piaget and Vygotsky. Because it is impossible to show with statistical significance that one type of manipulative benefits an entire age bracket, the research of educators must be combined with the research of child psychologists to suggest which types of manipulatives may be appropriate for various age groups based on success using the manipulative with small groups and general understanding of developmental cognitive ability.

Finally, another limitation to the research question is that different types of activities motivate different students. It is difficult to find a specific manipulative which an entire age group would find motivating to use because students are so different in personality and learning style. Because there is no universal strategy to motivate all students of a specific age group, the research can merely suggest manipulatives which motivate a majority of students.

Chapter 4: Results

In this chapter, the details of the results of the study are provided. For each age group, the suggested type of manipulatives to use to teach mathematics are introduced. The suggestions are based on student achievement, student engagement, and Common Core standards. A table summary of the results is given after the results. For definitions of the types of manipulatives see definitions in Appendix A and/or chapter 3, scope.

After an analysis of several studies regarding specific types of manipulatives, in conjunction with the research of cognitive ability and childhood development and alignment to Common Core, the following suggestions were found appropriate for each age group. The types of manipulatives are suggested for each age group based first on 1) if they were found beneficial to student achievement and motivation according to research studies, 2) if they aligned with the cognitive needs of the students based on research in child psychology and, 3) if they were aligned to the Common Core standards for the age group. If the type of manipulative did not meet any one of these three criteria it was not suggested for that age group. Table 4.1 summarizes the following results.

Kindergarten – Second Grade Suggestions

For the kindergarten through second grade age group, Physical Designated manipulatives and Virtual Designated manipulatives are the only forms of manipulatives found appropriate based on student achievement, student engagement, and Common Core standards.

Third – Fifth Grade Suggestions

For the third through fifth grade age group, Physical Designated, Physical Non-Designated, and Virtual Designated are the forms of manipulatives found appropriate based on student achievement, student engagement, and Common Core standards.

Sixth – Eighth Grade Suggestions

For the sixth through eighth grade age group, Physical Designated, Physical Non-Designated, Virtual Designated and Virtual Non-Designated are the forms of manipulatives found appropriate based on student achievement, student engagement, and Common Core standards.

Ninth – Twelfth Grade Suggestions

For the ninth through twelfth grade age group, Physical Non-Designated, Virtual Designated, and Virtual Non-Designated are the forms of manipulatives found appropriate based on students achievement, student engagement, and Common Core standards.

Table 4.1
Suggested Type of Manipulatives by Age Group

Age Group	*Type of Manipulative*			
	Physical Designated	Physical Non-Designated	Virtual Designated	Virtual Non-Designated
K-2 grade	Suggested	Not Suggested	Suggested	Not Suggested
3-5 grade	Suggested	Suggested	Suggested	Not Suggested
6-8 grade	Suggested	Suggested	Suggested	Suggested
9-12 grade	Not-Suggested	Suggested	Suggested	Suggested

Chapter 5: Analysis of Results

In this chapter, the results of the study are further analyzed and discussed. For each age group, the findings for the suggested type of manipulatives to use to teach mathematics are explained. Then, for each age group, the findings are analyzed and applied to the needs of modern educators. In the last section of the chapter, suggestions are provided for further research.

Kindergarten to Second Age Group

Explanation of Results

For the kindergarten through second grade age group, Physical Designated manipulatives and Virtual Designated manipulatives were the only forms of manipulatives suggested. First and foremost, the use of manipulatives must be designated by the teacher for students in this age group. According to Piaget, students in this age group would fall into the preoperational stage of development. In this stage, children begin using physical manipulatives to understand complex problem solving. Children may only be able to use manipulatives which are relatable and familiar to their background during this stage. Most importantly, a child in this stage may believe that when an object changes form, it is no longer the same object (Piaget, 1969). For example, a child might believe that a piece of paper is a different object once it is crumpled up into a ball. For this reason, the possibility of using non-designated manipulatives is not suggested for students in kindergarten through second grade.

In a research study conducted by Trisha Ainsa (1999), students between the ages of 3 and 6 years old were to compare the size of various geometric shapes.

Ainsa found there was improvement in geometric comparisons for both groups (using virtual and using physical manipulatives) with no difference between the groups. Ainsa's research also demonstrates the importance of the teacher guiding and designating the use of manipulatives for this age group.

David Brooks, Tammy Lyon, and Kent Steen conducted a research study with first grade students (2006). The students who were taught through the use of virtual manipulatives performed significantly better than the students in the control group. These results demonstrate the benefit of using virtual manipulatives to teach first grade geometry objectives. The treatment group teacher also observed the students' motivation and engagement in the lessons while using virtual manipulatives. The teacher reported that she believed her students "showed increased motivation and challenged themselves to higher levels." The findings of the treatment group teacher demonstrate the important role of engagement and motivation when using manipulatives at this age level.

In 1989, Evelyn Sowell compiled 60 studies regarding the effectiveness of using physical manipulatives to teach mathematics (not necessarily just geometry). Seventeen of the studies involved students in the kindergarten to second grade age bracket. Sowell found that at every age bracket, there was statistically significant evidence to show "that mathematics achievement is increased through the long-term use of concrete instructional materials and that students' attitudes toward mathematics are improved when they have instruction with concrete materials provided by teachers knowledgeable about their use." Not only does Sowell's analysis show that students' achievement in mathematics improved with the use of physical manipulatives but their motivation and engagement (noted in the analysis as attitude towards mathematics) was improved. Most importantly, Sowell's analysis supports the need for the teacher to designate the use of the manipulative. Sowell's analysis argues that if the teacher does not instruct on the use of the manipulative, the student will not experience the same type of success. While Sowell's analysis is applied to the teaching of mathematics in general, it can be applied to the importance of designating the use of manipulatives especially in the kindergarten through second grade age group.

Not only have the use of Physical Designated and Virtual Designated manipulatives been shown effective in improving student achievement and engagement through research studies and theories of child psychologists, the use of these types of manipulatives aligns with the Common Core Standards in geometry for this age group. The Common Core Standards in geometry for this age group are given in table 5.1.

Table 5.1
Common Core State Standards K-2

Kindergarten	CCSS.MATH.CONTENT.K.G.A.1- Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as above, below, beside, in front of, behind, and next to.
	CCSS.MATH.CONTENT.K.G.A.2- Correctly name shapes regardless of their orientations or overall size.
	CCSS.MATH.CONTENT.K.G.A.3- Identify shapes as two-dimensional (lying in a plan, "flat") or three-dimensional ("solid").
	CCSS.MATH.CONTENT.K.G.B.4- Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts and other attributes.
	CCSS.MATH.CONTENT.K.G.B.5- Model shapes in the world by building shapes from components and drawing shapes.
	CCSS.MATH.CONTENT.K.G.B.6- Compose simple shapes to form

	larger shapes.
First Grade	CCSS.MATH.CONTENT.1.G.A.1- Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.
	CCSS.MATH.CONTENT.1.G.A.2- Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.
	CCSS.MATH.CONTENT.1.G.A.3- Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.
Second Grade	CCSS.MATH.CONTENT.2.G.A.1- Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces. Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.
	CCSS.MATH.CONTENT.2.G.A.2- Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.
	CCSS.MATH.CONTENT.2.G.A.3- Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Each of the kindergarten standards refers to the use of some form of physical manipulative (in some cases virtual manipulative may apply) by the wording of the standard. Wording such as “describe objects in the environment”, “two-dimensional versus three-dimensional”, “model shapes in the world”, “building”, and “compose simple shapes to form larger shapes” allude to the need for manipulatives. For many of these standards, the manipulatives must be physical because they refer to three-dimensional geometry in real world scenarios. Other standards refer to comparing and combining shapes in two-dimensional planes. These manipulatives may be physical or virtual. Still with each of these standards, the role of the manipulative in learning must be clearly established by the teacher.

For the first grade standards, the role of the manipulatives is also clearly established. Wording such as "Compose two-dimensional shapes or three-dimensional shapes", "to create", "compose new shapes", and "partition" specifically call for the use of manipulatives to model how shapes are formed and how they relate to other shapes. Again, most of these standards seem to refer to physical manipulatives. However, given the proper virtual manipulatives, students could easily be shown how to model the same problems through software. The use of the manipulatives must be designated by the teacher in these standards as well. For example, the teacher will need to demonstrate how to partition a shape with the manipulatives before the student could use the manipulative properly.

The wording of the second grade standards further leads to the need for the use of manipulatives. Wording such as "recognize", "partition", "identical", and "need not have the same shape", call for comparisons that can be made through manipulatives. Unlike the kindergarten and first grade standards, the second grade standards could be taught through virtual and/or physical manipulatives because the modeling to be done through the manipulative is not as related to real-world scenarios. Still, the teacher will need to demonstrate the objective behind the use of the manipulative. For example, the teacher will need to teach the names of the shapes and how to identify them before the student can learn from the manipulative without guidance.

Kindergarten to Second Age Group

Analysis

Research and analysis suggests that, for the kindergarten through second grade age group, teachers use physical and/or virtual manipulatives to teach geometry as long as they designate how the manipulative is to be used. For geometry teachers, this helps to narrow down the choices of which manipulatives to use in their classroom and how to use them. Teachers of kindergarten, first grade, and second grade students should choose manipulatives that have a clear purpose. For example, a kindergarten teacher should choose cubes which can be stacked together to make a larger cube or a first grade teacher should choose a set of geometry tiles that fit together to form partitions of a larger shape. In both examples, the teacher will need to instruct the use of the manipulatives. Based on the research and analysis of the question, students in this age group will not benefit (in the categories of student achievement, student engagement, or alignment to Common Core Standards) from the use of manipulatives if they are not taught the specific role of the manipulative. In other words, if a student in this age group is given a manipulative without instruction as to how to use it, they will likely not use it to solve a complex geometry problem.

Third Grade to Fifth Grade Group

Explanation of Results

For the third to fifth grade group, the Physical Designated, Physical Non-Designated, and Virtual Designated types of manipulatives are suggested. Piaget (1969) explains that the majority of children in this age group are in the concrete operational stage of development. In this stage, children have a better understanding of concrete examples and can comprehend the symbolism of objects. At this age, most children are not able to visualize complex problems in their head. Because children in this stage have the cognitive ability to model complex problems with concrete examples, designating the use of manipulatives is useful as well as not-designating the use of manipulatives. However, because children may still struggle with visualizing complex problems without physical manipulatives, using non-designated virtual manipulatives is not suggested.

The analysis of Piaget's work is parallel to the findings of many imperical studies. Much like the kindergarten to second grade age group, Evelyn Sowell analyzed 17 studies regarding the use of physical manipulatives in teaching mathematics to students in the third grade to fifth grade age group. For this age group, Sowell also found with statistical significance that students' achievement in mathematics improved with the use of physical manipulatives and their motivation and engagement (noted in the analysis as attitude towards mathematics) was improved. She once again found that the students benefited from direct instruction of how to use the specific manipulatives. Sowell's analysis of these studies supports the suggestion that the third grade to fifth grade age group should be taught through the use of designated virtual and physical manipulatives.

In a study, conducted by Barbara Armstrong and Carol Larson, they recorded whether or not the fourth grade students utilized the manipulatives to solve the problem without being taught the use of the manipulative (1995). The majority of the fourth grade students did not utilize the manipulatives provided. After recording the results, the students were taught how to utilize the manipulatives to compare the area of different shapes. After being taught how to use the manipulatives, all twelve students preformed better on a set of similar shape area comparison questions. Armstrong and Larson's study is limited because it includes a very small sample. Therefore, the study does not prove with statistical significance that fourth grade students would preform better on area comparison problems when taught how to use manipulatives. However, Armstrong and Larson's study highlights an interesting point from Piaget's theory of the child's cognitive development at this age (Piaget, 1969). Piaget explained that children in this age range can understand the use of concrete manipulatives to represent complex ideas. The students in the study were capable of using the manipulatives, some of them required instruction on how to use them. For this reason, it is suggest that educators who teach geometry to students in this age range teach students using physical manipulatives that are designated and physical manipulatives that are not designated. It is important for educators of this particular age range to recognize to allow students the opportunity to explore the use of manipulatives without being taught how to use them.

Sinan Olkun conducted a study of fourth and fifth grade students (2003). The students were randomly placed into three different groups: a virtual manipulative treatment group, a physical manipulative treatment group, and a control group (no manipulatives). The students in the treatment group's test scores improved with statistical significance. For the fourth grade students, their scores improved more with the use of the physical manipulatives. For the fifth grade students, their scores improved more with the virtual manipulatives. Olkun's study demonstrates not only the benefit of virtual and physical manipulatives on student achievement (in this case the students score on the post-test) but also demonstrates how engagement is related to type of manipulative for students in this age group. Because students in the fifth grade improved more with the virtual manipulatives and students in the fourth grade improved more with the physical manipulative, it is reasonable to believe the type of manipulative plays a role in student motivation. Perhaps fifth grade students are more engaged when using virtual manipulatives and fourth grade students are more engaged when using physical manipulatives.

The use of Physical Designated, Physical Non-Designated, and Virtual Designated manipulatives in teaching geometry to this age group also aligns with the Common Core Standards in geometry for this age group. The Common Core Standards in Geometry for this age group are displayed in Table 5.2 on the next page.

Table 5.2

Common Core State Standards 3rd-5th

Third Grade	CCSS.MATH.CONTENT.3.G.A.1- Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.
	CCSS.MATH.CONTENT.3.G.A.2- Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.
Fourth Grade	CCSS.MATH.CONTENT.4.G.A.1- Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.
	CCSS.MATH.CONTENT.4.G.A.2- Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.
	CCSS.MATH.CONTENT.4.G.A.3- Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.
Fifth Grade	CCSS.MATH.CONTENT.5.G.A.1- Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., *x*-axis and *x*-coordinate, *y*-axis and *y*-coordinate).
	CCSS.MATH.CONTENT.5.G.A.2- Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.
	CCSS.MATH.CONTENT.5.G.B.3- Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.
	CCSS.MATH.CONTENT.5.G.B.4- Classify two-dimensional figures in a hierarchy based on properties.

Throughout all of these standards, the wording strongly suggests the need for manipulatives. Wording such as "recognize shapes", "draw examples", "partition shapes", "identify", "classify", "can be folded", "represent", "understand attributes", and "classify" imply the need for physical or virtual manipulatives. Still, for most of these standards, the use of the manipulative must be designated because the purpose of the manipulatives' use needs to be taught in order to achieve the objective of the standard. For example, before a student can use the physical manipulative of paper shapes in half to understand symmetry, the student must be taught how and why folding the paper in half represents symmetry.

Third Grade to Fifth Grade Group

Analysis

Research and analysis suggests for the third grade through fifth grade age group, teachers should use physical and/or virtual manipulatives to teach geometry. Teachers should designate how the student uses some of the physical manipulatives and how they use all of the virtual manipulatives. Students in this age group are capable of solving complex problems through concrete manipulatives with and without guidance from a teacher. However, students in this age group do not have the cognitive ability to use virtual manipulatives with no instruction because they have difficulty visualizing abstract/complex problems without concrete tools. For geometry teachers, this helps to narrow down the choices of which manipulatives to use in their classroom and how to use them. Teachers of third grade, fourth grade, and fifth grade students should encourage students to use all types of physical manipulatives. Teachers should guide the students' use of the manipulatives to meet specific objectives but they should also allow the students to explore complex problems through the use of physical manipulatives without designating the use of the manipulative. Virtual manipulatives have proved very effective for this age group when the teacher instructs the use of the manipulative. Because virtual manipulatives are slightly more abstract for the average cognitive ability of a child in this age group, using non-designated virtual manipulatives is generally not recommended.

Sixth Grade to Eighth Grade Group

Explanation of Results

For the sixth grade to eighth grade group, all four types of manipulatives are suggested. Piaget (1969) explained that around age twelve (sixth grade) children enter the formal operational stage of development. Children in this stage begin to develop the ability to visualize an abstract/complex problem without the use of manipulatives. Still, a child in this stage will still need to be exposed to manipulatives of some form because the child may visualize a particular manipulative to understand a problem. In order to visualize the manipulative, the child must be exposed to it. Children in this stage are no longer limited by their inability to visualize complex ideas in geometry. Therefore, there is no limitation as to which type of manipulatives they might learn from using.

With Piaget's understanding of a child's cognitive development in this age group, it is no surprise empirical studies, such as Evelyn Sowell's, Trisha Ainsa, and

Barbara Armstrong and Carol Larson (as previously mentioned), demonstrate that both physical and virtual manipulatives benefit the achievement of a child in this age group as compared to the grade level average (Sowell, 1989; Ainsa, 1999; Armstrong and Larson, 1995).

In an article for Mathematics Teaching in the Middle School, Dana Weiss compiled evidence from several studies regarding the use of manipulatives in sixth, seventh and eighth grade classrooms (2006). According to Weiss, very little research exists regarding the use of any type of manipulative in teaching geometry to this age group. Weiss hypothesized the lack of research to this end is due to the fact that many teachers do not utilize manipulatives with students older than the fifth grade age group. Many manipulatives are toy-like in nature or serve to teach the very basic concepts of geometry. For sixth, seventh, and eighth grade math teachers, it is difficult to justify the use of manipulatives because they believe their students are not interested in using them. From the little research Weiss synthesized, manipulatives were beneficial to students overall understanding of complex geometric concepts. Weiss also explains that students in this age group should be allowed the use of manipulatives without being designated by a teacher. Weiss warns that the teacher may need to guide the use of some manipulatives. Still, in regards to the use of other manipulatives, students should be allowed to use them to make their own discoveries about the complexities of geometry.

The use of all four types of manipulatives in teaching geometry to this age group also aligns with the Common Core Standards in geometry for this age group. The Common Core Standards in geometry for this age group are given in Table 5.3.

Table 5.3

Common Core State Standards 6th -8th

Sixth Grade	CCSS.MATH.CONTENT.6.G.A.1- Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
	CCSS.MATH.CONTENT.6.G.A.2- Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = l\ w\ h$ and $V = b\ h$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.
	CCSS.MATH.CONTENT.6.G.A.3- Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.
	CCSS.MATH.CONTENT.6.G.A.4- Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.
Seventh Grade	CCSS.MATH.CONTENT.7.G.A.1- Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.
	CCSS.MATH.CONTENT.7.G.A.2- Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.
	CCSS.MATH.CONTENT.7.G.A.3- Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.
	CCSS.MATH.CONTENT.7.G.B.4- Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.
	CCSS.MATH.CONTENT.7.G.B.5- Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.
	CCSS.MATH.CONTENT.7.G.B.6- Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons,

	cubes, and right prisms.
Eighth Grade	CCSS.MATH.CONTENT.8.G.A.1- Verify experimentally the properties of rotations, reflections, and translations. CCSS.MATH.CONTENT.8.G.A.2- Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.
	CCSS.MATH.CONTENT.8.G.A.3- Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.
	CCSS.MATH.CONTENT.8.G.A.4- Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.
	CCSS.MATH.CONTENT.8.G.A.5- Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles.
	CCSS.MATH.CONTENT.8.G.B.6- Explain a proof of the Pythagorean Theorem and its converse.
	CCSS.MATH.CONTENT.8.G.B.7- Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
	CCSS.MATH.CONTENT.8.G.B.8- Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.
	CCSS.MATH.CONTENT.8.G.B.9- Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

In the sixth grade standards, there are specific objectives which include the use of manipulatives. Not all of the standards explicitly say that a virtual or a physical manipulative must be used. In CCSS.MATH.CONTENT.6.G.A.1, students are expected to be able to compose or decompose the shape of rectangle by using triangles. Then, students are expected to apply this to a real-world scenario. In order to demonstrate how a rectangle is made of two triangles, the student must use a manipulative (either virtual or physical) to show the composition and decomposition. Similarly, CCSS.MATH.CONTENT.6.G.A.2 calls for the specific use of unit cube manipulatives. CCSS.MATH.CONTENT.6.G.A.3 and CCSS.MATH.CONTENT.6.G.A.4 call for the use of manipulatives because of wording such as “real-world” and “represent”. However, both of these objectives could be taught through either a virtual or a physical manipulative.

In the seventh grade standards, a greater emphasis is placed on three-dimensional geometry. The standards expect students to be able to explain three-dimensional geometry using two-dimensional geometry. To understand the relationship between two-dimensional geometry, standards such as CCSS.MATH.CONTENT.7.G.A.3 and CCSS.MATH.CONTENT.7.G.B.6 suggest the use of manipulatives (not explicitly virtual or physical). Allowing the students to use manipulatives without instruction may help them to discover the relationship between two-dimensional and three-dimensional geometry and develop their own deeper understanding of the relationship.

The first set of eighth standards (8.G.A.1-5) are complex extensions of CCSS.MATH.CONTENT.8.G.A.1. In this standard, students are to “verify experimentally” the concepts of rotations, reflections, and translations. In other words, students must use manipulatives (either physical or virtual) to how a shape changes when rotated, reflected, and translated. In the case of this standard, the teacher should not designate the use of the manipulative because the standard implies for the student to experiment on his/her own.

Sixth Grade to Eighth Grade Group

Analysis

Research and analysis suggests for the sixth grade through eigth grade age group, teachers should use physical and/or virtual manipulatives to teach geometry. Students in this age group are capable of solving complex problems through concrete manipulatives with and without guidance from a teacher. Students in this age group are also developing their ability to visualize complex geometry problems. Consequently, they are able to use virtual manipulatives with and without guidance from the teacher. Despite common belief of modern educators, manipulatives are very useful for teaching geometry to this age group. The use of any of the four types of manipulatives could improve student achievement and student engagement. For sixth through eighth grade geometry teachers, this means there is no need to narrow down the type of manipulatives used in the classroom. Teachers need only determine which concepts their students might benefit from learning through manipulatives. The type of manipulative should not be a primary concern as long as it applies to the concept. The teacher should also allow students the opportunity to use manipulatives with and without a designated use.

Ninth Grade to Twelfth Grade Group

Explanation of Results

For the ninth grade to twelfth grade group, non-designated physical manipulatives are suggested, Virtual Designated manipulatives and Virtual Non-Designated manipulatives are suggested. Similar to the sixth through eighth grade group, the ninth grade to twelfth grade age group are in the concrete operational stage of cognitive development according to Piaget (1969). Therefore, students in this age groups have the ability to visualize complex/abstract geometry concepts without manipulatives. Still, manipulatives are beneficial to assisting students in this age group to develop the ability to visualize concepts.

For reasons similar to those described by Weiss (2006) in her research regarding the use of manipulatives in sixth, seventh and eighth grade classrooms, very little research exists regarding the use of manipulatives in ninth through twelfth grade geometry classrooms. Still, the research which does exist argues that manipulatives benefit student achievement and engagement for ninth through twelfth grade students. Andreja Starcic, Mara Cotic, and Matej Zajc conducted a study of ninth through twelfth grade age students in Britain (2013). Students were not tested to determine if achievement was increased. Instead, students were asked if they felt their understanding of the concepts had improved because of the software they were provided. Nearly all of the students felt the software had improved their understanding. The students also found the software to be engaging. Starcic, Cotic, and Zajc's study does not statistically prove the benefit of virtual manipulatives to student achievement, however, it provides sufficient evidence to refute the common belief among educators that manipulatives are not relatable to ninth through twelfth grade students. In fact, the success of the software program, argues for the support of both Virtual Designated and Virtual Non-Designated manipulatives in the ninth to twelfth grade classroom.

For the ninth through twelfth grade age group, the Common Core Standards are grouped together. The standards are also more complex and numerous than for the other grade groups. Still, some standards imply the use of manipulatives. The following standards for this age group imply the need for the use of manipulatives:

CCSS.MATH.CONTENT.HSG.CO.A.2- Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

CCSS.MATH.CONTENT.HSG.CO.A.5- Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

CCSS.MATH.CONTENT.HSG.CO.D.12- Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

CCSS.MATH.CONTENT.HSG.CO.D.13- Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Each of these standards explicitly requires the use of either a virtual or a physical manipulative. In the case of the physical manipulatives, the standards imply that the student should explore how to represent, explain, or construct various geometric concepts. Therefore, it is not suggested that geometry teachers use Physical Designated manipulatives because the use of that type of manipulative does not align with Common Core standards. Furthermore, there is no evidence to suggest that Physical Designated manipulatives are beneficial to either student achievement or student engagement. From the wording of the Common Core Standards, the evidence from the few empirical studies, and the explanation of Piaget about the cognitive ability of children in ninth through twelfth grade, the types of manipulatives suggested are Physical Non-Designated, Virtual Designated, and Virtual Non-Designated.

Ninth Grade to Twelfth Grade Group

Analysis

Research and analysis suggests for the ninth grade through twelfth grade age group, teachers should use Physical Non-Designated and/or virtual manipulatives to teach geometry. Students in this age group are capable of solving complex problems through concrete manipulatives with and without guidance from a teacher. Students in this age group are also developing their ability to visualize complex geometry problems. Consequently, they are able to use virtual manipulatives with and without guidance from the teacher. Despite common belief of modern educators, manipulatives are very useful for teaching geometry to this age group. The use of virtual manipulatives, designated or not, is believed to improve student engagement for this age group. While there is not a great deal of empirical prove proving the benefit of using manipulatives with this age group, the Common Core standards align with the use of these three types of manipulatives. For teachers, this means students should be allowed time to use manipulatives. Students in this age groups are not developmentally beyond using manipulatives.

Suggestions for Further Research

To research the use of manipulatives in teaching geometry further, more empirical studies could be conducted to test the benefit of various types of manipulatives on student achievement with different groups. Another area for further research would be to investigate the benefits of using manipulatives with different groups of students. For example, do physical or virtual manipulatives benefit students with learning disabilities? Further research might also include more studies about student engagement. For example, qualitative research of how students respond to using manipulatives in the classroom.

APPENDIX I
DEFINITIONS

<u>Student Achievement-</u> The student's ability to demonstrate an understanding of a concept as compared to normal grade level performance.

<u>Student Engagement-</u> The student's motivation to learn based on interest, curiosity, self-efficacy, and focus as related to the how the material is taught.

<u>Physical Non-Designated Manipulatives-</u> Hands on tools provided to students where the teacher does not guide how the manipulative is to be used to solve the problem.

<u>Physical Designated Manipulatives-</u> Hands on tools provided to students where the teacher guides how the manipulative is to be used to solve the problem.

<u>Virtual Non-Designated Manipulatives-</u> Hands on tools through computer software where the teacher does not guide how the manipulative is to be used to solve the problem.

<u>Virtual Designated Manipulatives-</u> Hands on tools through computer software where the teacher guides how the manipulative is to be used to solve the problem.

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